MATHEMATICS

Part - II

STANDARD X

The Coordination Committee formed by GR No. Abhyas - 2116/(Pra.Kra.43/16) SD - 4 Dated 25.4.2016 has given approval to prescribe this textbook in its meeting held on 29.12.2017 and it has been decided to implement it from the educational year 2018-19.

Mathematics

Part II

STANDARD X

Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune - 411 004

PTLHEL

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First Edition : 2018

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Paper

70 GSM Cream wove

Print Order No.

N/PB/2018-19/50,000

Printer

DELIGHT PRINTERS, PUNE

Publisher

Vivek Uttam Gosavi, Controller
Maharashtra State Textbook Bureau,
Prabhadevi Mumbai- 400 025

The Constitution of India

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

NATIONAL ANTHEM

Jana-gana-mana-adhināyaka jaya hē
Bhārata-bhāgya-vidhātā,

Panjāba-Sindhu-Gujarāta-Marāthā
Drāvida-Utkala-Banga

Vindhya-Himāchala-Yamunā-Gangā
uchchala-jaladhi-taranga

Tava subha nāmē jāgē, tava subha āsisa māgē,
gāhē tava jaya-gāthā,

Jana-gana-mangala-dāyaka jaya hē
Bhārata-bhāgya-vidhātā,

Jaya hē, Jaya hē, Jaya hē,
Jaya jaya jaya, jaya hē.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect, and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness.

Preface

Dear Students,

Welcome to the tenth standard !

This year you will study two text books - Mathematics Part-I and Mathematics Part-II

The main areas in the book Mathematics part - II are Geometry, Trigonometry, Coordinate geometry and Mensuration. All of these topics were introduced in the ninth standard. This year you will study some more details of the same. Their utility will be clear from the given examples. Wherever a new unit, formula or application is introduced, its lucid explanation is given. Each chapter contains illustrative solved examples and sets of questions for practice. Moreover, some questions in practice sets are star-marked, indicating that they are challenging for talented students.

After Tenth standard, some students do not opt for mathematics. They too need the basic concepts and the knowledge necessary for working in other fields.The matter under the head 'For more Information' is useful for those students who wish to study mathematics after tenth standard and achieve proficiency in it. So they are earnestly advised to study this part. Read the book thoroughly at least once and grasp the concepts.

Additional audio visual material regarding each lesson will be available to you by Q.R. Code through 'App'. It will definitely be useful to you for your studies.

Much importance is given to the tenth standard examination. You are advised not to take the stress and study to the best of your ability to achieve expected success.

Best wishes for it !

(Dr. Sunil Magar)
Director
Maharashtra State Bureau of Textbook
Production and Curriculum Research, Pune.

Pune
Date : 18 March 2018, Gudhipadva
Indian Solar Year : 27 Falgun 1939

It is expected that students will develop the following competencies after studying Mathematics- Part II syllabus in standard X

Area	Topic	Competency Statements
1. Geometry	1.1 Similar triangles 1.2 Circle	The students will be able to - • solve examples using properties of similar trianlges, properties of congruent traingles and Pythagoras theorem. • construct similar triangles. • be able to use properties of chords and tangents. • be able to construct tangents to a circle.
2. Co-ordinate Geometry	2.1 Co-ordinate geometry	• find distance between two points. • find the co-ordinates of a point dividing a segment in given ratio. • find slope of a line.
3. Mensuration	3.1 Surface area and volume	• find length of arc of a circle. • find areas of sector of a circle and segment of a circle. • compute surface areas and volumes of some three dimensional objects.
4. Trigonometry	4.1 Trigonometry	• solve examples using trigionometric identities • solve problems like measuring height of a tree, width of a river bed etc., using trigonometry.

Instructions for Teachers

Read the book in detail and grasp the content thoroughly.Take the help of activities to explain different topics, to verify the formulae etc.

Practicals is also a means of evaluation. Activities given can be used for this purpose. Encourage the students to think independently.Compliment a student if he solves an example by a different and logically correct method.

Suitable activities, other than those given in the text book, can be planned to understand the statements of the theorems and to develop the skill to solve problems.

List of some practicals (Specimen)

1. Cut out a triangular piece of card-board. Place a lit up candle or a small lamp on a table. Hold the triangle between a wall and the candle/ lamp. Observe the shadow of the triangle. Decide if the triangle and its shadow are similar. (What care will you take so that the triangle and its shadow are similar?)
2. Cut out two identical right angled triangles. Name the vertices of the triangles as A, B, C on both sides. Draw the altitude on the hypotenuse of one of them. Name the foot of the perpendicular as D. Cut the triangle on its altitude and obtain two triangles. State the correspondences by which the three triangles are similar with one another.
3. Draw a circle. Take three points - one on the circle, one in its interior and one in its exterior. Prepare a table, showing rough figures and stating how many tangents can be drawn to the circle through each of the three points.
4. Draw at least five different circles passing through two given distinct points indicating that innumerable circles can be drawn passing through them.
5. Take a geoboard on which nails are suitably fixed to verify properties of a circle. Prepare a figure using rubber bands for any one of the following theorems.

   (i) Inscribed angle theorem (ii) Tangent secant theorem of angles

   (iii) Theorem of angles inscribed in opposite arcs of a circle.
6. Prepare a model of a circle and an angle. Show different arcs intercepted by the angle in different situations. Draw the corresponding figures in your note book.
7. Draw an angle and divide it into four equal parts using compass and ruler.
8. Take a beaker. Measure its height and radius of base. Calculate its capacity using the formula. Fill it fully with water. Measure the volume of the water with a measuring cylinder. Compare the two results and draw inference.
9. Take a paper cup of the shape of frustum of a cone. Measure the radii of its base and top and also its height. Using formula, calculate its capacity. Fill it fully with water and then measure the volume of the water. Compare the measured and the calculated volumes and verify the formula.
10. Cut two similar triangles out of a card-board. Decide by actual measurements -

    (i) Are their areas proportional to the squares of their perimeters ?

    (ii) Are their areas proportional to the squares of their medians ?

INDEX

1 Similarity

Let's study.

- Ratio of areas of two triangles
- Basic proportionality theorem
- Converse of basic proportionality theorem
- Tests of similarity of triangles
- Property of an angle bisector of a triangle
- Property of areas of similar triangles
- The ratio of the intercepts made on the transversals by three parallel lines

Let's recall.

We have studied Ratio and Proportion. The statement, 'the numbers a and b are in the ratio $\frac{m}{n}$' is also written as, 'the numbers a and b are in proportion m:n.'
For this concept we consider positive real numbers. We know that the lengths of line segments and area of any figure are positive real numbers.
We know the formula of area of a triangle.

Area of a triangle = $\frac{1}{2}$ Base × Height

Let's learn.

Ratio of areas of two triangles

Let's find the ratio of areas of any two triangles.

Ex. In Δ ABC, AD is the height and BC is the base.
In Δ PQR, PS is the height and QR is the base

$$\frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{\frac{1}{2} \times BC \times AD}{\frac{1}{2} \times QR \times PS}$$

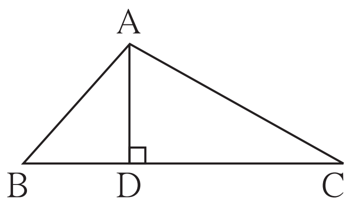


Fig. 1.1

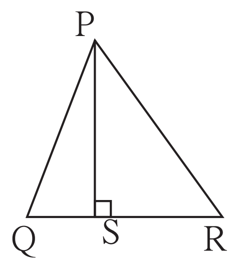


Fig. 1.2

$$\therefore \frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{BC \times AD}{QR \times PS}$$

Hence the ratio of the areas of two triangles is equal to the ratio of the products of their bases and corresponding heights.

Base of a triangle is b_1 and height is h_1. Base of another triangle is b_2 and height is h_2 . Then the ratio of their areas = $\frac{b_1 \times h_1}{b_2 \times h_2}$

Suppose some conditions are imposed on these two triangles,

Condition 1 : If the heights of both triangles are equal then-

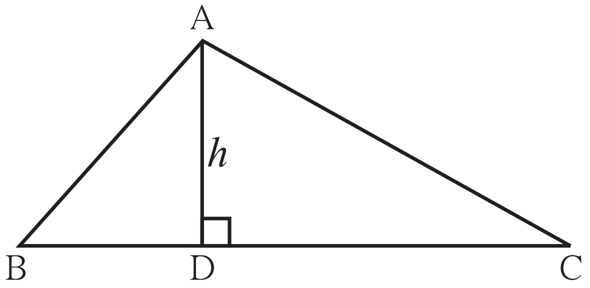


Fig. 1.3

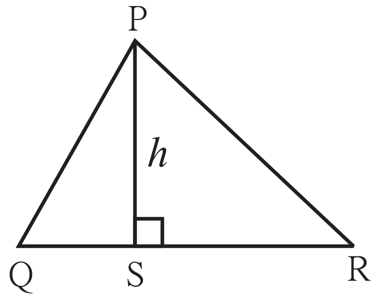


Fig. 1.4

$$\frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{BC \times h}{QR \times h} = \frac{BC}{QR}$$

$$\therefore \frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{b_1}{b_2}$$

Property: The ratio of the areas of two triangles with equal heights is equal to the ratio of their corresponding bases.

Condition 2 : If the bases of both triangles are equal then -

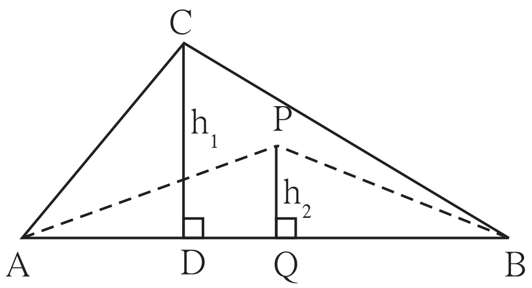


Fig. 1.5

$$\frac{A(\Delta ABC)}{A(\Delta APB)} = \frac{AB \times h_1}{AB \times h_2}$$

$$\frac{A(\Delta ABC)}{A(\Delta APB)} = \frac{h_1}{h_2}$$

Property: The ratio of the areas of two triangles with equal bases is equal to the ratio of their corresponding heights.

Activity :

Fill in the blanks properly.

(i)

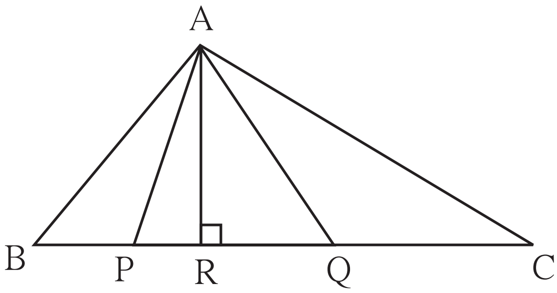


Fig. 1.6

$$\frac{A(\Delta ABC)}{A(\Delta APQ)} = \frac{\square \times \square}{\square \times \square} = \frac{\square}{\square}$$

(ii)

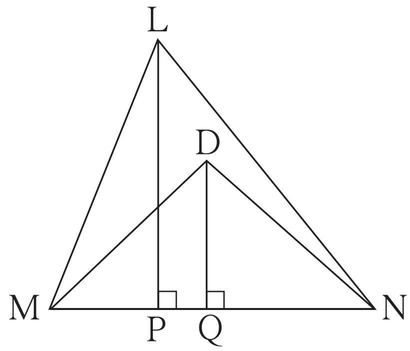


Fig.1.7

$$\frac{A(\Delta LMN)}{A(\Delta DMN)} = \frac{\square \times \square}{\square \times \square} = \frac{\square}{\square}$$

(iii) M is the midpoint of seg AB and seg CM is a median of Δ ABC

$$\therefore \frac{A(\Delta AMC)}{A(\Delta BMC)} = \frac{\square}{\square}$$

$$= \frac{\square}{\square} = \square$$

State the reason.

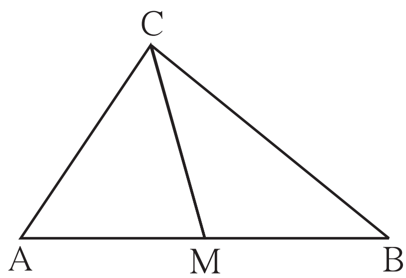


Fig. 1.8

Solved Examples

Ex. (1)

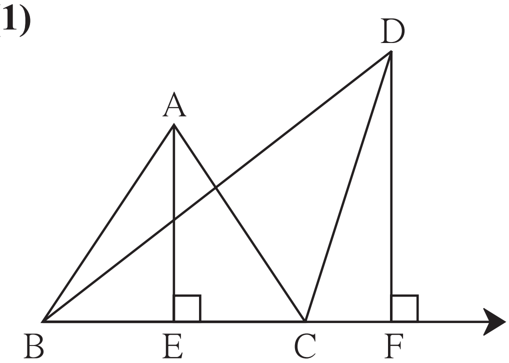


Fig.1.9

In adjoining figure

AE ⊥ seg BC, seg DF ⊥ line BC,

AE = 4, DF = 6 , then find $\frac{A(\Delta ABC)}{A(\Delta DBC)}$.

Solution : $\frac{A(\Delta ABC)}{A(\Delta DBC)} = \frac{AE}{DF}$ .... bases are equal, hence areas proportional to heights.

$= \frac{4}{6} = \frac{2}{3}$

Ex. (2) In Δ ABC point D on side BC is such that DC = 6, BC = 15. Find A(Δ ABD) : A(Δ ABC) and A(Δ ABD) : A(Δ ADC) .

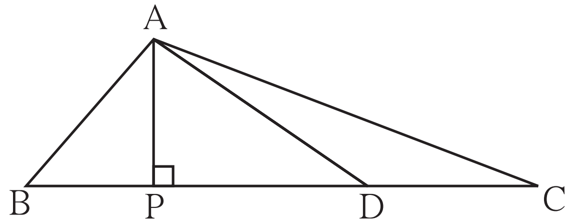


Fig. 1.10

Solution : Point A is common vertex of Δ ABD, Δ ADC and Δ ABC and their bases are collinear. Hence, heights of these three triangles are equal

BC = 15, DC = 6 $\therefore$ BD = BC − DC = 15 − 6 = 9

$\frac{A(\Delta ABD)}{A(\Delta ABC)} = \frac{BD}{BC}$ .... heights equal, hence areas proportional to bases.

$= \frac{9}{15} = \frac{3}{5}$

$\frac{A(\Delta ABD)}{A(\Delta ADC)} = \frac{BD}{DC}$ .... heights equal, hence areas proportional to bases.

$= \frac{9}{6} = \frac{3}{2}$

Ex. (3)

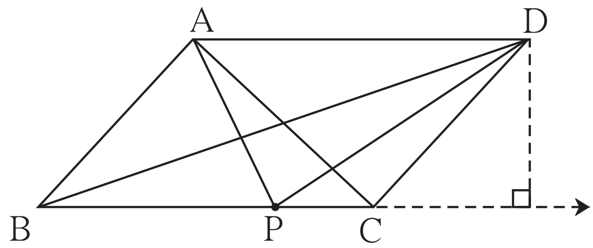


Fig. 1.11

$\square$ ABCD is a parallelogram. P is any point on side BC. Find two pairs of triangles with equal areas.

Solution : $\square$ ABCD is a parallelogram.

$\therefore$ AD || BC and AB || DC

Consider Δ ABC and Δ BDC.

Both the triangles are drawn in two parallel lines. Hence the distance between the two parallel lines is the height of both triangles.

In Δ ABC and Δ BDC, common base is BC and heights are equal.

Hence, A(Δ ABC) = A(Δ BDC)

In Δ ABC and Δ ABD, AB is common base and and heights are equal.

$\therefore$ A(Δ ABC) = A(Δ ABD)

Ex.(4)

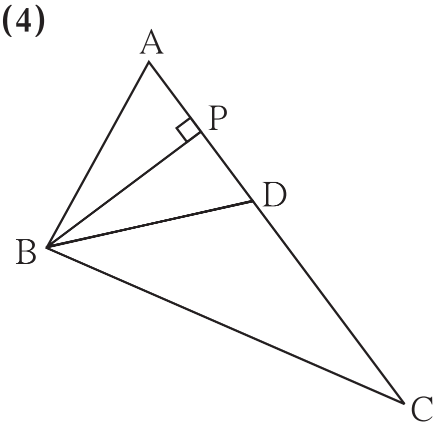


Fig. 1.12

In adjoining figure in Δ ABC, point D is on side AC. If AC = 16, DC = 9 and BP $\perp$ AC, then find the following ratios.

(i) $\dfrac{A(\Delta ABD)}{A(\Delta ABC)}$ (ii) $\dfrac{A(\Delta BDC)}{A(\Delta ABC)}$

(iii) $\dfrac{A(\Delta ABD)}{A(\Delta BDC)}$

Solution : In Δ ABC point P and D are on side AC, hence B is common vertex of Δ ABD, Δ BDC, Δ ABC and Δ APB and their sides AD, DC, AC and AP are collinear. Heights of all the triangles are equal. Hence, areas of these triangles are proportinal to their bases. AC = 16, DC = 9

$\therefore$ AD = 16 − 9 = 7

$\therefore \dfrac{A(\Delta ABD)}{A(\Delta ABC)} = \dfrac{AD}{AC} = \dfrac{7}{16}$ . . . . triangles having equal heights

$\dfrac{A(\Delta BDC)}{A(\Delta ABC)} = \dfrac{DC}{AC} = \dfrac{9}{16}$ . . . . triangles having equal heights

$\dfrac{A(\Delta ABD)}{A(\Delta BDC)} = \dfrac{AD}{DC} = \dfrac{7}{9}$ . . . . triangles having equal heights

Remember this!

- Ratio of areas of two triangles is equal to the ratio of the products of their bases and corresponding heights.
- Areas of triangles with equal heights are proportional to their corresponding bases.
- Areas of triangles with equal bases are proportional to their corresponding heights.

Practice set 1.1

1. Base of a triangle is 9 and height is 5. Base of another triangle is 10 and height is 6. Find the ratio of areas of these triangles.

2. In figure 1.13 BC ⊥ AB, AD ⊥ AB,

BC = 4, AD = 8, then find $\frac{A(\Delta ABC)}{A(\Delta ADB)}$.

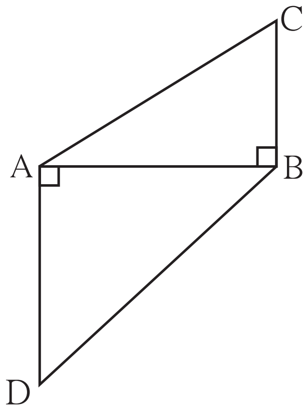


Fig. 1.13

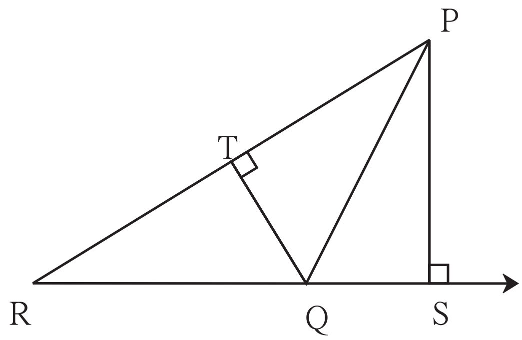


Fig. 1.14

3. In adjoining figure 1.14

seg PS ⊥ seg RQ seg QT ⊥ seg PR.

If RQ = 6, PS = 6 and PR = 12,

then find QT.

4. In adjoining figure, AP ⊥ BC,

AD || BC, then find

A(Δ ABC) : A(Δ BCD).

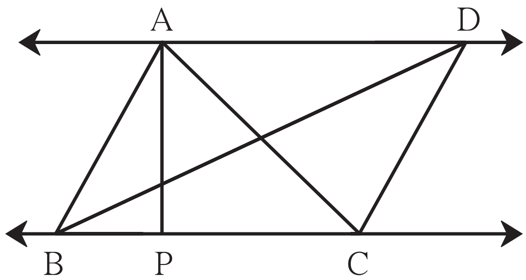


Fig. 1.15

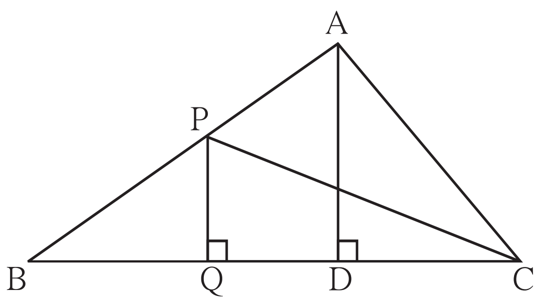


Fig. 1.16

5. In adjoining figure PQ ⊥ BC,

AD ⊥ BC then find following ratios.

(i) $\frac{A(\Delta PQB)}{A(\Delta PBC)}$ (ii) $\frac{A(\Delta PBC)}{A(\Delta ABC)}$

(iii) $\frac{A(\Delta ABC)}{A(\Delta ADC)}$ (iv) $\frac{A(\Delta ADC)}{A(\Delta PQC)}$

Let's learn.

Basic proportionality theorem

Theorem : **If a line parallel to a side of a triangle intersects the remaining sides in two distinct points, then the line divides the sides in the same proportion.**

Given : In Δ ABC line $l \parallel$ line BC and line l intersects AB and AC in point P and Q respectively

To prove : $\frac{AP}{PB} = \frac{AQ}{QC}$

Construction: Draw seg PC and seg BQ

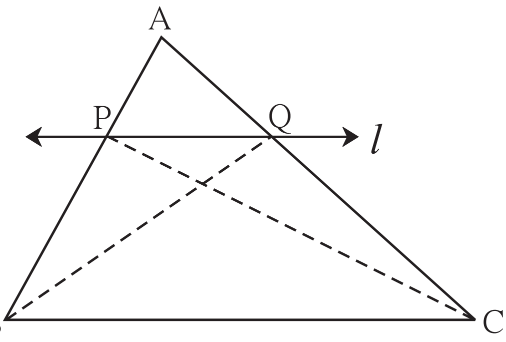


Fig. 1.17

Proof : Δ APQ and Δ PQB have equal heights.

$\therefore \frac{A(\Delta APQ)}{A(\Delta PQB)} = \frac{AP}{PB}$ .... (I) (areas proportionate to bases)

and $\frac{A(\Delta APQ)}{A(\Delta PQC)} = \frac{AQ}{QC}$ .... (II) (areas proportionate to bases)

seg PQ is common base of Δ PQB and Δ PQC. seg PQ $\parallel$ seg BC,

hence Δ PQB and Δ PQC have equal heights.

$A(\Delta PQB) = A(\Delta PQC)$ .... (III)

$\frac{A(\Delta APQ)}{A(\Delta PQB)} = \frac{A(\Delta APQ)}{A(\Delta PQC)}$ .... [from (I), (II) and (III)]

$\therefore \frac{AP}{PB} = \frac{AQ}{QC}$ .... [from (I) and (II)]

Converse of basic proportionality theorem

Theorem : **If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.**

In figure 1.18, line l interesects the side AB and side AC of Δ ABC in the points P and Q respectively and $\frac{AP}{PB} = \frac{AQ}{QC}$, hence line $l \parallel$ seg BC.

This theorem can be proved by indirect method.

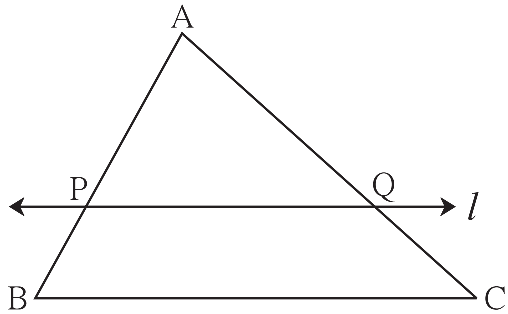


Fig. 1.18

Activity :

- Draw a Δ ABC.
- Bisect $\angle$ B and name the point of intersection of AC and the angle bisector as D.
- Measure the sides.

  AB = $\square$ cm BC = $\square$ cm

  AD = $\square$ cm DC = $\square$ cm
- Find ratios $\frac{AB}{BC}$ and $\frac{AD}{DC}$.
- You will find that both the ratios are almost equal.
- Bisect remaining angles of the triangle and find the ratios as above. You can verify that the ratios are equal.

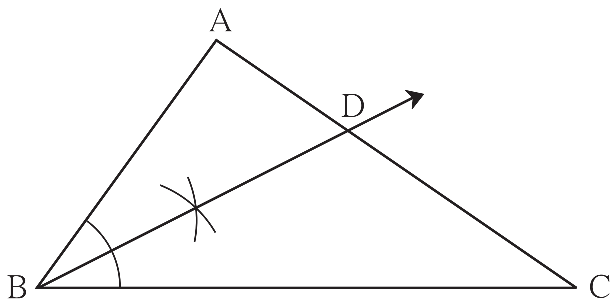


Fig. 1.19

Let's learn.

Property of an angle bisector of a triangle

Theorem : The bisector of an angle of a triangle divides the side opposite to the angle in the ratio of the remaining sides.

Given : In Δ ABC, bisector of $\angle$C interesects seg AB in the point E.

To prove : $\frac{AE}{EB} = \frac{CA}{CB}$

Construction : Draw a line parallel to ray CE, passing through the point B. Extend AC so as to intersect it at point D.

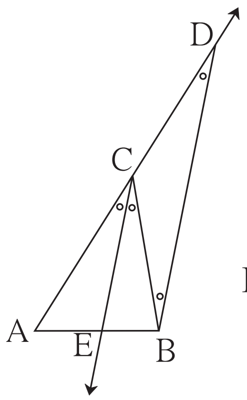


Fig. 1.20

Proof : ray CE || ray BD and AD is transversal,

$\therefore \angle ACE = \angle CDB$ .... (corresponding angles) ...(I)

Now taking BC as transversal

$\angle ECB = \angle CBD$ .... (alternate angle) ...(II)

But $\angle ACE \cong \angle ECB$ .... (given) ...(III)

$\therefore \angle CBD \cong \angle CDB$ .... [from (I), (II) and (III)]

In Δ CBD, side CB $\cong$ side CD ....(sides opposite to congruent angles)

$\therefore$ CB = CD ...(IV)

Now in Δ ABD, seg EC || seg BD .... (construction)

$\therefore \frac{AE}{EB} = \frac{AC}{CD}$ ....(Basic proportionality theorem)..(V)

$\therefore \frac{AE}{EB} = \frac{AC}{CB}$ .... [from (IV) and (V)]

For more information :

Write another proof of the theorem yourself.

Draw DM $\perp$ AB and DN $\perp$ AC. Use the following properties and write the proof.

(1) The areas of two triangles of equal heights are proportional to their bases.

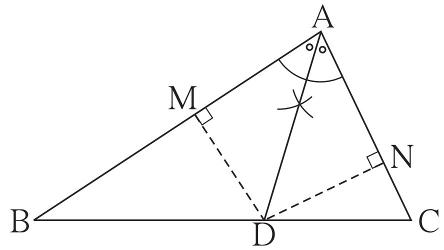


Fig. 1.21

(2) Every point on the bisector of an angle is equidistant from the sides of the angle.

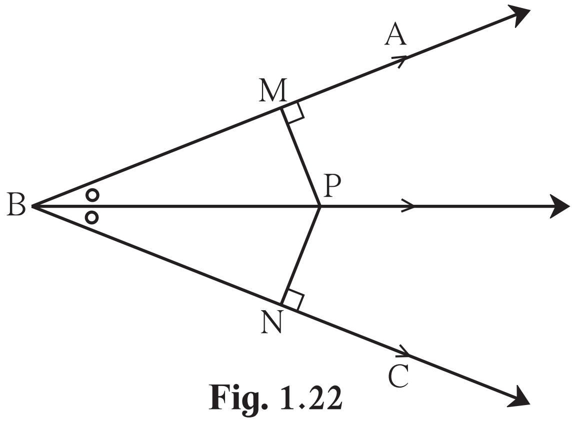


Fig. 1.22

Converse of angle bisector theorem

If in Δ ABC, point D on side BC such that $\frac{AB}{AC} = \frac{BD}{DC}$, then ray AD bisects $\angle$ BAC.

Property of three parallel lines and their transversals

Activity:

- Draw three parallel lines.
- Label them as *l*, *m*, *n*.
- Draw transversals t_1 and t_2.
- AB and BC are intercepts on transversal t_1.
- PQ and QR are intercepts on transversal t_2.

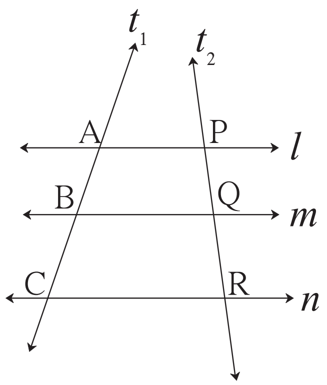


Fig. 1.23

- Find ratios $\frac{AB}{BC}$ and $\frac{PQ}{QR}$. You will find that they are almost equal.

Theorem : The ratio of the intercepts made on a transversal by three parallel lines is equal to the ratio of the corrosponding intercepts made on any other transversal by the same parallel lines.

Given : line *l* || line *m* || line *n*

t_1 and t_2 are transversals.

Transversal t_1 intersects the lines in points A, B, C and t_2 intersects the lines in points P, Q, R.

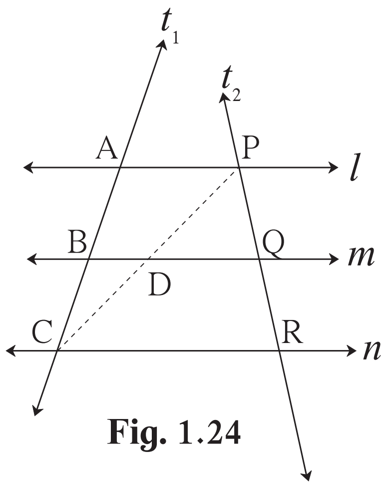


Fig. 1.24

To prove : $\frac{AB}{BC} = \frac{PQ}{QR}$

Proof : Draw seg PC , which intersects line *m* at point D.

In Δ ACP, BD || AP

$\therefore \frac{AB}{BC} = \frac{PD}{DC}$. . . . . (I) (Basic proportionality theorem)

In Δ CPR, DQ || CR

$\therefore \frac{PD}{DC} = \frac{PQ}{QR}$. . . . . (II) (Basic proportionality theorem)

$\therefore \frac{AB}{BC} = \frac{PD}{DC} = \frac{PQ}{QR}$. . . . . . from (I) and (II).

$\therefore \frac{AB}{BC} = \frac{PQ}{QR}$

Remember this !

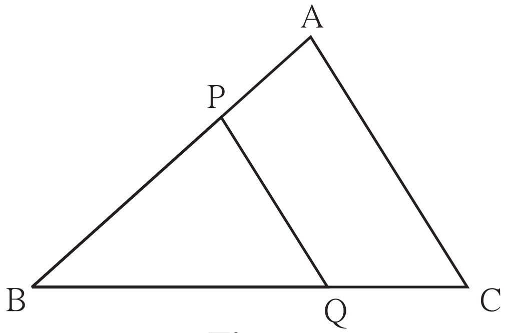


Fig. 1.25

(1) Basic proportionality theorem.

In Δ ABC, if seg PQ || seg AC

then $\frac{AP}{BP} = \frac{QC}{BQ}$

(2) Converse of basic proportionality theorem.

In ΔPQR, if $\frac{PS}{SQ} = \frac{PT}{TR}$

then seg ST || seg QR.

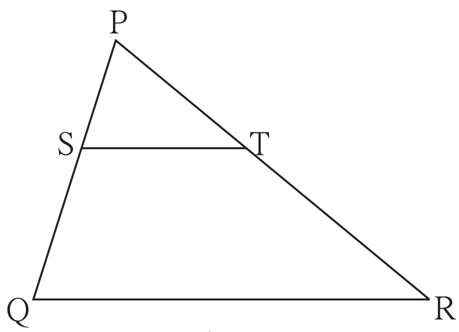


Fig. 1.26

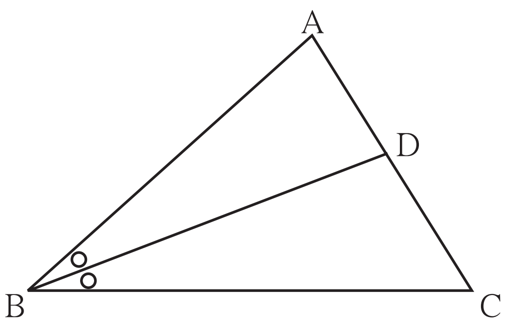


Fig. 1.27

(3) Theorem of bisector of an angle of a triangle.

If in Δ ABC, BD is bisector of $\angle$ ABC,

then $\frac{AB}{BC} = \frac{AD}{DC}$

(4) Property of three parallel lines and their transversals.

If line AX || line BY || line CZ and line l and line m are their transversals then $\frac{AB}{BC} = \frac{XY}{YZ}$

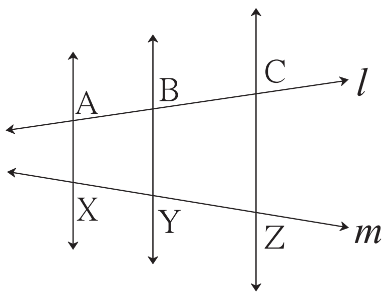


Fig. 1.28

Solved Examples

Ex. (1) In Δ ABC, DE || BC

If DB = 5.4 cm, AD = 1.8 cm

EC = 7.2 cm then find AE.

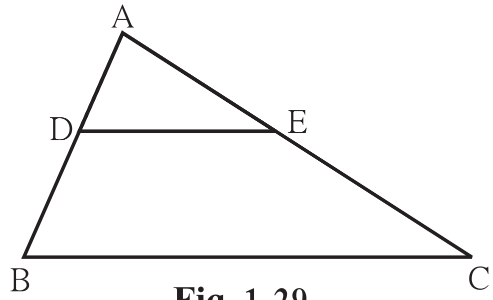


Fig. 1.29

Solution : In Δ ABC, DE || BC

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ ..... Basic proportionality theorem

$\therefore \frac{1.8}{5.4} = \frac{AE}{7.2}$

$\therefore$ AE $\times$ 5.4 = 1.8 $\times$ 7.2

$\therefore AE = \frac{1.8 \times 7.2}{5.4} = 2.4$

AE = 2.4 cm

Ex. (2) In Δ PQR, seg RS bisects $\angle$ R.

If PR = 15, RQ = 20 PS = 12

then find SQ.

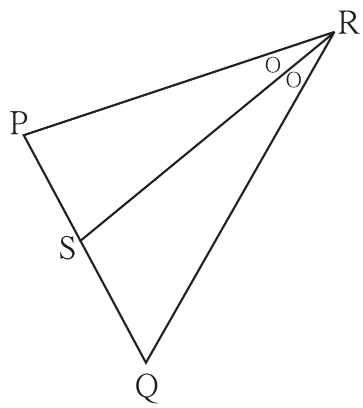


Fig. 1.30

Solution : In Δ PRQ, seg RS bisects $\angle$ R.

$\frac{PR}{RQ} = \frac{PS}{SQ}$ . . . . . . property of angle bisector

$\frac{15}{20} = \frac{12}{SQ}$

$SQ = \frac{12 \times 20}{15}$

$\therefore$ SQ = 16

Activity :

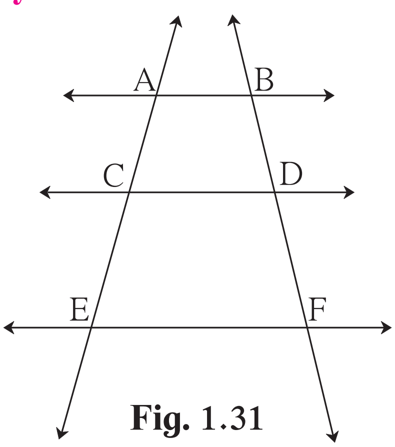


Fig. 1.31

In the figure 1.31, AB || CD || EF

If AC = 5.4, CE = 9, BD = 7.5

then find DF

Solution : AB || CD || EF

$\frac{AC}{\square} = \frac{\square}{DF}$ . . . . . . (   )

$\frac{5.4}{9} = \frac{\square}{DF}$ $\therefore$ DF = $\square$

Activity :

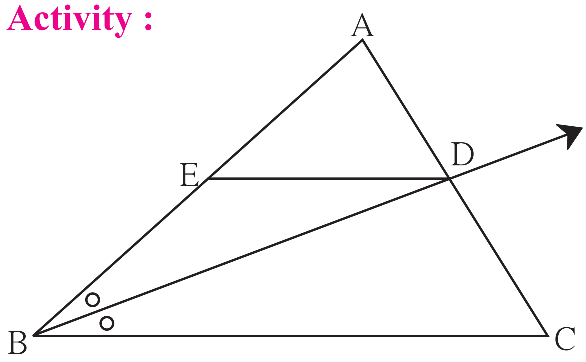


Fig. 1.32

In ΔABC, ray BD bisects $\angle$ABC. A–D–C, side DE || side BC, A–E–B then prove that, $\frac{AB}{BC} = \frac{AE}{EB}$

Proof : In Δ ABC, ray BD bisects $\angle$ B.

$\therefore \frac{AB}{BC} = \frac{AD}{DC}$ . . (I)(Angle bisector theorem)

In Δ ABC, DE || BC

$\frac{AE}{EB} = \frac{AD}{DC}$ . . . . . (II) (. . . . .)

$\frac{AB}{\square} = \frac{\square}{EB}$ . . . . . . from (I) and (II)

Practice set 1.2

1. Given below are some triangles and lengths of line segments. Identify in which figures, ray PM is the bisector of $\angle$ QPR.

(1)

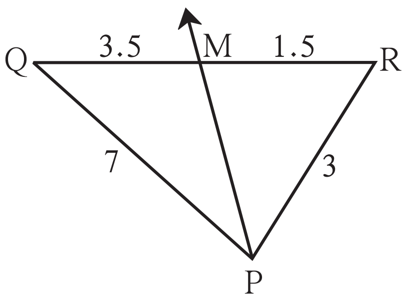


Fig. 1.33

(2)

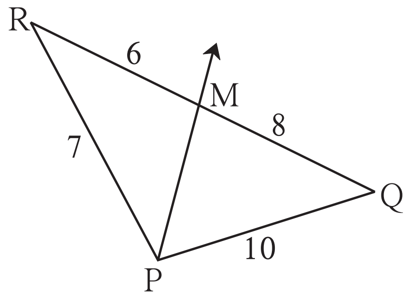


Fig. 1.34

(3)

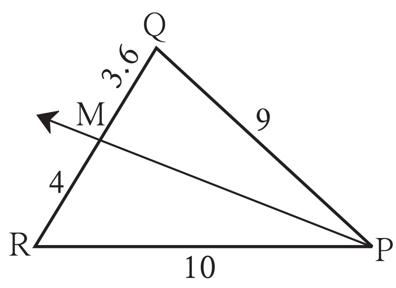


Fig. 1.35

2. In Δ PQR, PM = 15, PQ = 25 PR = 20, NR = 8. State whether line NM is parallel to side RQ. Give reason.

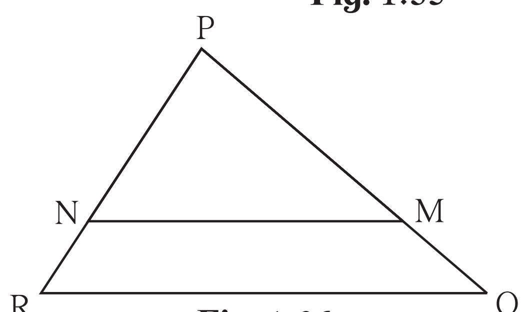


Fig. 1.36

3. In Δ MNP, NQ is a bisector of $\angle$ N. If MN = 5, PN = 7 MQ = 2.5 then find QP .

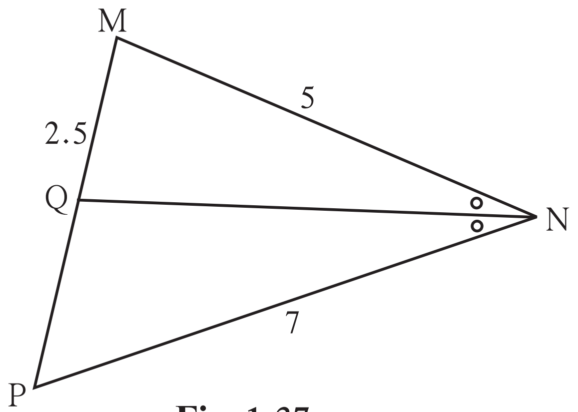


Fig. 1.37

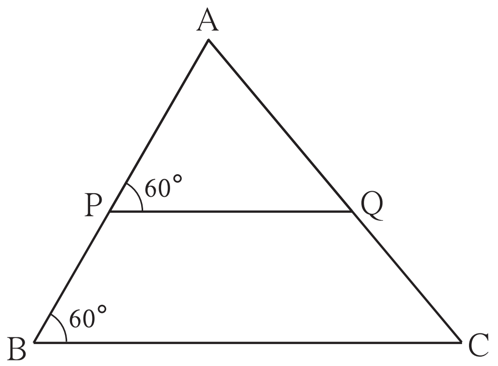


Fig. 1.38

4. Measures of some angles in the figure are given. Prove that

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

5. In trapezium ABCD, side AB || side PQ || side DC, AP = 15, PD = 12, QC = 14, find BQ.

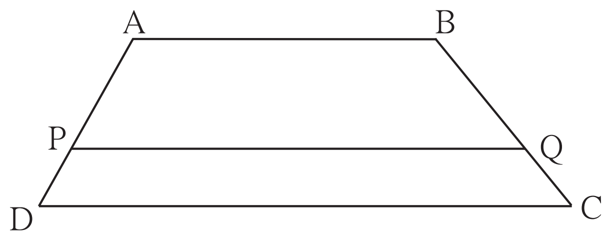


Fig. 1.39

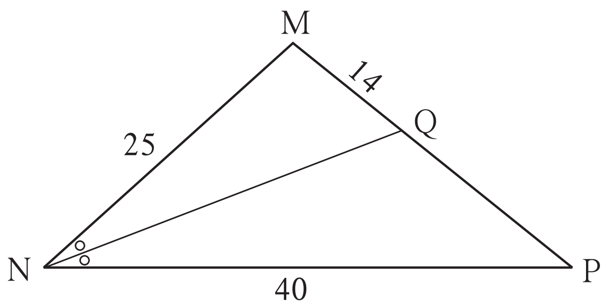


Fig. 1.40

6. Find QP using given information in the figure.

7. In figure 1.41, if AB || CD || FE then find x and AE.

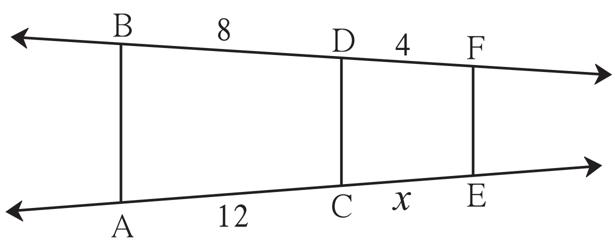


Fig. 1.41

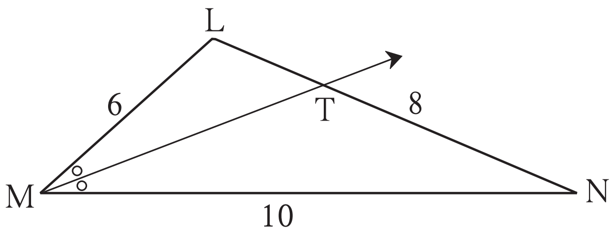


Fig. 1.42

8. In Δ LMN, ray MT bisects ∠ LMN If LM = 6, MN = 10, TN = 8, then find LT.

9. In Δ ABC, seg BD bisects ∠ ABC. If AB = x, BC = $x + 5$, AD = $x - 2$, DC = $x + 2$, then find the value of x.

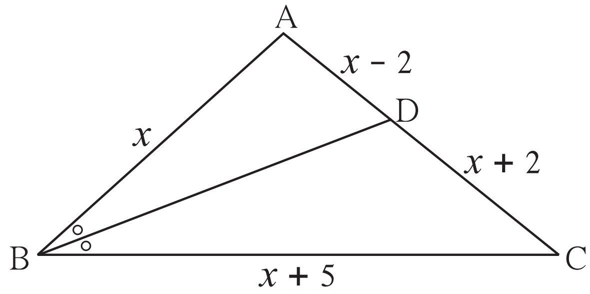


Fig. 1.43

10. In the figure 1.44, X is any point in the interior of triangle. Point X is joined to vertices of triangle. Seg PQ || seg DE, seg QR || seg EF. Fill in the blanks to prove that, seg PR || seg DF.

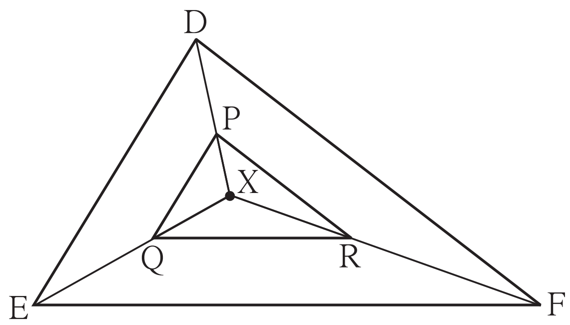


Fig. 1.44

Proof : In Δ XDE, PQ || DE .... ____

∴ $\frac{XP}{\_\_\_\_} = \frac{\_\_\_\_}{QE}$ .... (I) (Basic proportionality theorem)

In Δ XEF, QR || EF .... ____

∴ $\frac{\_\_\_\_}{\_\_\_\_} = \frac{\_\_\_\_}{\_\_\_\_}$ ....(II) ____

∴ $\frac{\_\_\_\_}{\_\_\_\_} = \frac{\_\_\_\_}{\_\_\_\_}$ .... from (I) and (II)

∴ seg PR || seg DE .... (converse of basic proportionality theorem)

11★. In Δ ABC, ray BD bisects ∠ ABC and ray CE bisects ∠ ACB. If seg AB ≅ seg AC then prove that ED || BC.

Let's recall.

Similar triangles

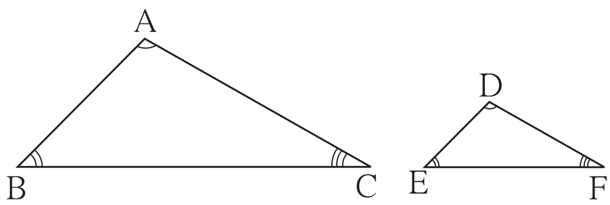


Fig. 1.45

In Δ ABC and Δ DEF, if $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$

and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

then ΔABC and ΔDEF are similar triangles.

'Δ ABC and Δ DEF are similar' is expressed as '$\Delta ABC \sim \Delta DEF$'

Let's learn.

Tests of similarity of triangles

For similarity of two triangles, the necessary conditions are that their corresponding sides are in same proportion and their corresponding angles are congruent. Out of these conditions; when three specifc conditions are fulfilled, the remaining conditions are automatically fulfilled. This means for similarity of two triangles, only three specific conditions are sufficient. Similarity of two triangles can be confirmed by testing these three conditions. The groups of such sufficient conditions are called tests of similarity, which we shall use.

AAA test for similarity of triangles

For a given correspondence of vertices, when corresponding angles of two triangles are congruent, then the two triangles are similar.

In Δ ABC and Δ PQR, in the correspondence $ABC \leftrightarrow PQR$ if $\angle A \cong \angle P$, $\angle B \cong \angle Q$ and $\angle C \cong \angle R$ then $\Delta ABC \sim \Delta PQR$.

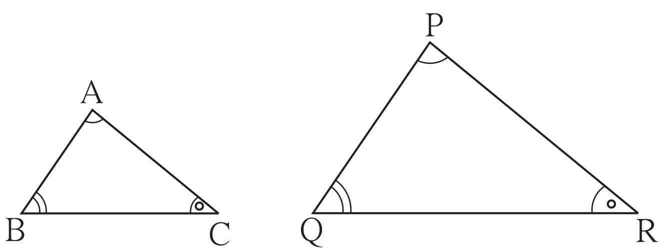


Fig. 1.46

For more information :

Proof of AAA test

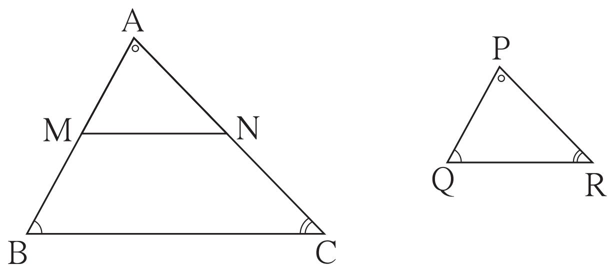


Fig. 1.47

Given : In Δ ABC and Δ PQR,
$\angle A \cong \angle P$, $\angle B \cong \angle Q$,
$\angle C \cong \angle R$.

To prove : Δ ABC ~ Δ PQR

Let us assume that Δ ABC is bigger than Δ PQR. Mark point M on AB, and point N on AC such that AM = PQ and AN = PR.

Show that Δ AMN $\cong$ Δ PQR. Hence show that MN $\parallel$ BC.

Now using basic proportionality theorem, $\frac{AM}{MB} = \frac{AN}{NC}$

That is $\frac{MB}{AM} = \frac{NC}{AN}$ .... (by invertendo)

$\frac{MB+AM}{AM} = \frac{NC+AN}{AN}$ .... (by componendo)

$\therefore \frac{AB}{AM} = \frac{AC}{AN}$

$\therefore \frac{AB}{PQ} = \frac{AC}{PR}$

Similarly it can be shown that $\frac{AB}{PQ} = \frac{BC}{QR}$

$\therefore \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ $\quad\therefore$ Δ ABC ~ Δ PQR

A A test for similarity of triangles:

We know that for a given correspondence of vertices, when two angles of a triangle are congruent to two corresponding angles of another triangle, then remaining angle of first triangle is congruent to the remaining angle of the second triangle.

This means, when two angles of one triangle are congruent to two corresponding angles of another triangle then this condition is sufficient for similarity of two triangles. This condition is called AA test of similarity.

SAS test of similarity of triangles

For a given correspondence of vertices of two triangles, if two pairs of corresponding sides are in the same proportion and the angles between them are congruent, then the two triangles are similar.

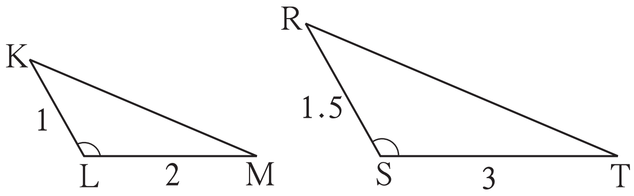


Fig. 1.48

For example, if in Δ KLM and Δ RST,

$\angle$ KLM $\cong$ $\angle$ RST

$$\frac{KL}{RS} = \frac{LM}{ST} = \frac{2}{3}$$

Therefore, Δ KLM $\sim$ Δ RST

SSS test for similarity of triangles

For a given correspondence of vertices of two triangles, when three sides of a triangle are in proportion to corresponding three sides of another triangle, then the two triangles are similar.

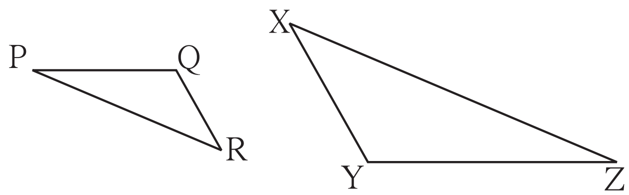


Fig. 1.49

For example, if in Δ PQR and Δ XYZ,

If $\frac{PQ}{YZ} = \frac{QR}{XY} = \frac{PR}{XZ}$

then Δ PQR $\sim$ Δ ZYX

Properties of similar triangles :

(1) Δ ABC $\sim$ Δ ABC - Reflexivity

(2) If Δ ABC $\sim$ Δ DEF then Δ DEF $\sim$ Δ ABC - Symmetry

(3) If Δ ABC $\sim$ Δ DEF and Δ DEF $\sim$ Δ GHI, then Δ ABC $\sim$ Δ GHI - Transitivity

Solved Examples

Ex. (1) In Δ XYZ,

$\angle Y = 100°$, $\angle Z = 30°$,

In Δ LMN,

$\angle M = 100°$, $\angle N = 30°$,

Are Δ XYZ and Δ LMN similar? If yes, by which test?

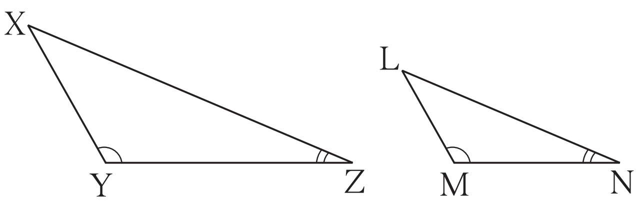


Fig. 1.50

Solution : In Δ XYZ and Δ LMN,

$\angle Y = 100°$, $\angle M = 100°$, $\therefore \angle Y \cong \angle M$

$\angle Z = 30°$, $\angle N = 30°$, $\therefore \angle Z \cong \angle N$

$\therefore \Delta XYZ \sim \Delta LMN$ ..... by AA test.

Ex. (2) Are two triangles in figure 1.51 similar, according to the information given? If yes, by which test?

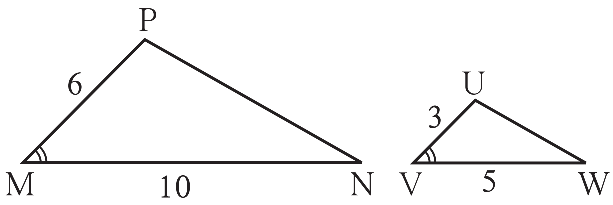


Fig. 1.51

Solution : In Δ PMN and Δ UVW

$$\frac{PM}{UV} = \frac{6}{3} = \frac{2}{1}, \frac{MN}{VW} = \frac{10}{5} = \frac{2}{1}$$

$$\therefore \frac{PM}{UV} = \frac{MN}{VW}$$

and $\angle M \cong \angle V$ .... Given

$\Delta PMN \sim \Delta UVW$ .... SAS test of similarity

Ex. (3) Can we say that the two triangles in figure 1.52 similar, according to information given? If yes, by which test ?

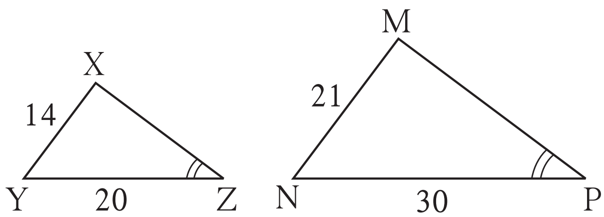


Fig. 1.52

Solution : Δ XYZ and Δ MNP,

$$\frac{XY}{MN} = \frac{14}{21} = \frac{2}{3},$$

$$\frac{YZ}{NP} = \frac{20}{30} = \frac{2}{3}$$

It is given that $\angle Z \cong \angle P$.

But $\angle Z$ and $\angle P$ are not included angles by sides which are in proportion.

$\therefore \Delta$ XYZ and Δ MNP can not be said to be similar.

Ex. (4) In the adjoining figure $BP \perp AC$, $CQ \perp AB$, A – P– C, A– Q– B , then prove that Δ APB and Δ AQC are similar.

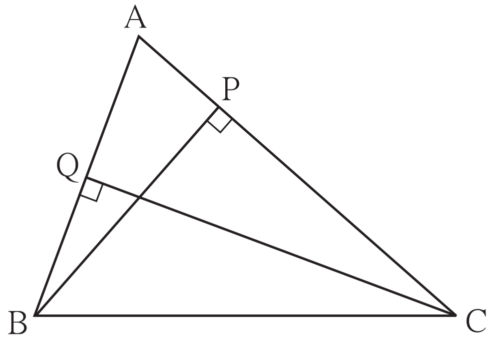


Fig. 1.53

Solution : In Δ APB and Δ AQC

$\angle$ APB = $\square$° (I)

$\angle$ AQC = $\square$° (II)

$\therefore$ $\angle$ APB $\cong$ $\angle$ AQC ....from(I) and (II)

$\angle$ PAB $\cong$ $\angle$ QAC .... ($\square$)

$\therefore$ Δ APB ~ Δ AQC ..... AA test

Ex. (5) Diagonals of a quadrilateral ABCD intersect in point Q. If 2QA = QC , 2QB = QD, then prove that DC = 2AB.

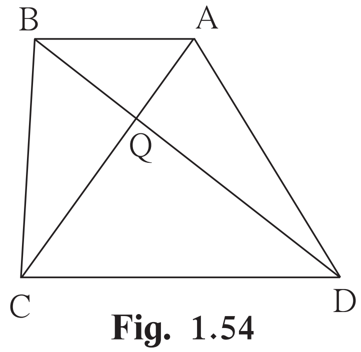


Fig. 1.54

Given : 2QA = QC

2QB = QD

To prove : CD = 2AB

Proof : 2QA = QC $\therefore \frac{QA}{QC} = \frac{1}{2}$ .... (I)

2QB = QD $\therefore \frac{QB}{QD} = \frac{1}{2}$ .... (II)

$\therefore \frac{QA}{QC} = \frac{QB}{QD}$ ....from (I) and (II)

In Δ AQB and Δ CQD,

$\frac{QA}{QC} = \frac{QB}{QD}$ .... proved

$\angle$ AQB $\cong$ $\angle$ DQC .... opposite angles

$\therefore$ Δ AQB ~Δ CQD .... (SAS test of similarity)

$\therefore \frac{AQ}{CQ} = \frac{QB}{QD} = \frac{AB}{CD}$ .... correponding sides are proportional

But $\frac{AQ}{CQ} = \frac{1}{2}$ $\therefore \frac{AB}{CD} = \frac{1}{2}$

$\therefore$ 2AB = CD

Practice set 1.3

1. In figure1.55, $\angle ABC = 75^\circ$, $\angle EDC = 75^\circ$ state which two triangles are similar and by which test? Also write the similarity of these two triangles by a proper one to one correspondence.

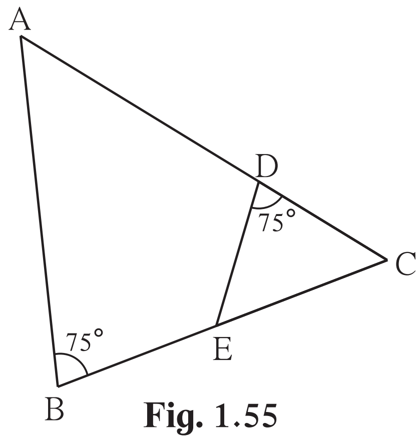


Fig. 1.55

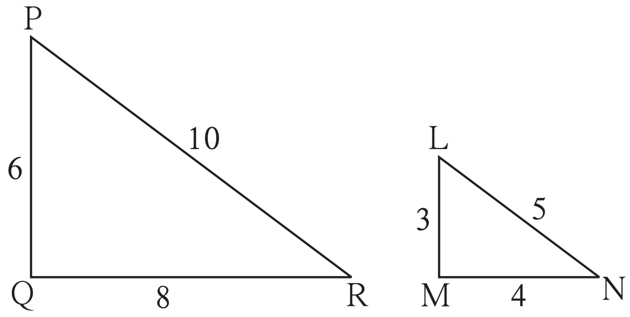


Fig. 1.56

2. Are the triangles in figure 1.56 similar? If yes, by which test ?

3. As shown in figure 1.57, two poles of height 8 m and 4 m are perpendicular to the ground. If the length of shadow of smaller pole due to sunlight is 6 m then how long will be the shadow of the bigger pole at the same time ?

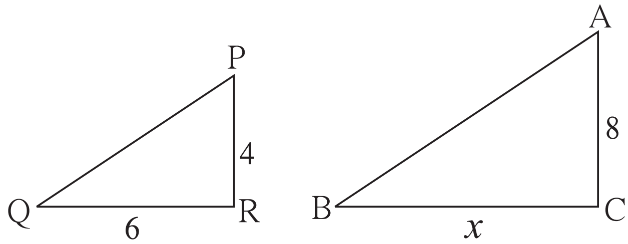


Fig. 1.57

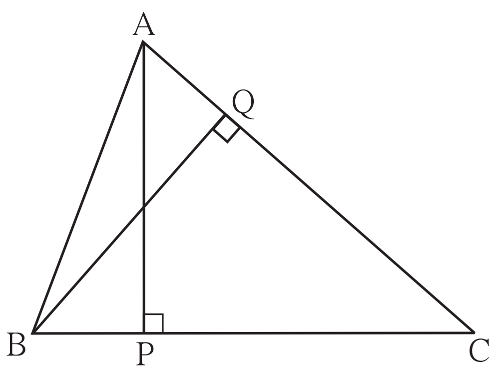


Fig. 1.58

4. In Δ ABC, AP $\perp$ BC, BQ $\perp$ AC B– P–C, A–Q – C then prove that, $\Delta CPA \sim \Delta CQB$.

If AP = 7, BQ=8, BC=12 then find AC.

5. **Given :** In trapezium PQRS, side PQ || side SR, AR = 5AP, AS = 5AQ then prove that, SR = 5PQ

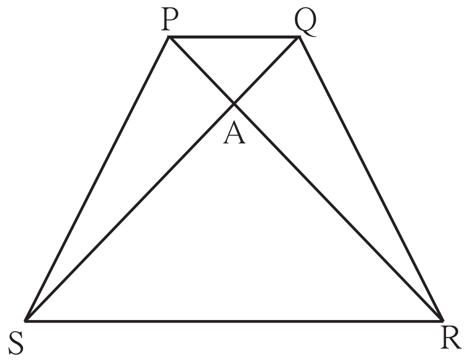


Fig. 1.59

6. In trapezium ABCD, (Figure 1.60) side AB || side DC, diagonals AC and BD intersect in point O. If AB = 20, DC = 6, OB = 15 then find OD.

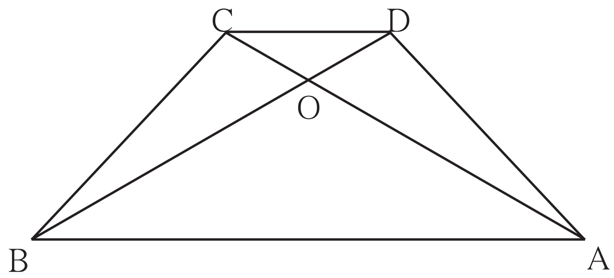


Fig. 1.60

7. □ABCD is a parallelogram point E is on side BC. Line DE intersects ray AB in point T. Prove that DE × BE = CE × TE.

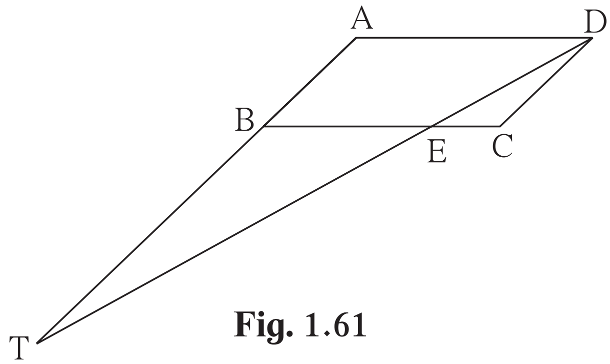


Fig. 1.61

8. In the figure, seg AC and seg BD intersect each other in point P and $\frac{AP}{CP} = \frac{BP}{DP}$. Prove that, Δ ABP ~ Δ CDP

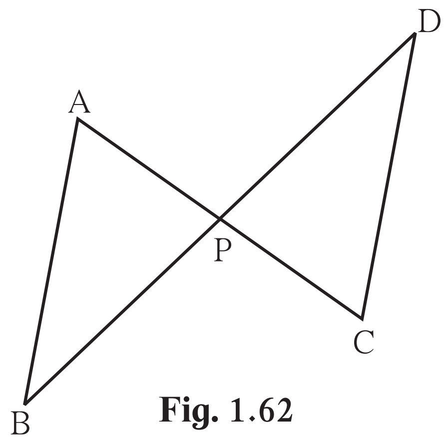


Fig. 1.62

9. In the figure, in Δ ABC, point D on side BC is such that, $\angle$ BAC = $\angle$ ADC. Prove that, $CA^2 = CB \times CD$

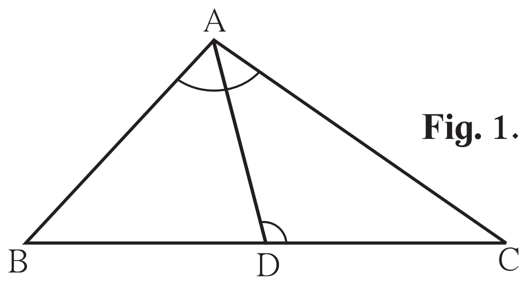


Fig. 1.63

Let's learn.

Theorem of areas of similar triangles

Theorem : When two triangles are similar, the ratio of areas of those triangles is equal to the ratio of the squares of their corresponding sides.

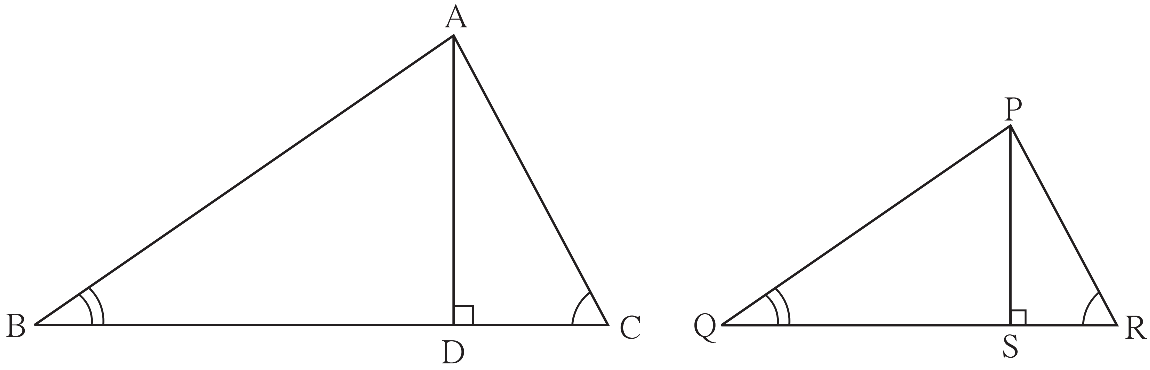


Fig. 1.64

Given : $\Delta ABC \sim \Delta PQR$, $AD \perp BC$, $PS \perp QR$

To prove: $\dfrac{A(\Delta ABC)}{A(\Delta PQR)} = \dfrac{AB^2}{PQ^2} = \dfrac{BC^2}{QR^2} = \dfrac{AC^2}{PR^2}$

Proof : $\dfrac{A(\Delta ABC)}{A(\Delta PQR)} = \dfrac{BC \times AD}{QR \times PS} = \dfrac{BC}{QR} \times \dfrac{AD}{PS}$ .... (I)

In ΔABD and ΔPQS,

$\angle B = \angle Q$ .... given

$\angle ADB = \angle PSQ = 90°$

$\therefore$ According to AA test $\Delta ABD \sim \Delta PQS$

$\therefore \dfrac{AD}{PS} = \dfrac{AB}{PQ}$ .... (II)

But $\Delta ABC \sim \Delta PQR$

$\therefore \dfrac{AB}{PQ} = \dfrac{BC}{QR}$ .... (III)

From (I), (II) and (III)

$$\dfrac{A(\Delta ABC)}{A(\Delta PQR)} = \dfrac{BC}{QR} \times \dfrac{AD}{PS} = \dfrac{BC}{QR} \times \dfrac{BC}{QR} = \dfrac{BC^2}{QR^2} = \dfrac{AB^2}{PQ^2} = \dfrac{AC^2}{PR^2}$$

Solved Examples

Ex. (1) : $\Delta ABC \sim \Delta PQR$, A (ΔABC) = 16 , A (ΔPQP) = 25, then find the value of ratio $\frac{AB}{PQ}$.

Solution : $\Delta ABC \sim \Delta PQR$

$$\therefore \frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{AB^2}{PQ^2}$$ .... theorem of areas of similar triangles

$$\therefore \frac{16}{25} = \frac{AB^2}{PQ^2} \quad \therefore \frac{AB}{PQ} = \frac{4}{5}$$ .... taking square roots

Ex. (2) Ratio of corresponging sides of two similar triangles is 2:5, If the area of the small triangle is 64 sq.cm. then what is the area of the bigger triangle ?

Solution : Assume that $\Delta ABC \sim \Delta PQR$.

Δ ABC is smaller and Δ PQR is bigger triangle.

$$\therefore \frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{(2)^2}{(5)^2} = \frac{4}{25}$$ .... ratio of areas of similar triangles

$$\therefore \frac{64}{A(\Delta PQR)} = \frac{4}{25}$$

$4 \times A(\Delta PQR) = 64 \times 25$

$$A(\Delta PQR) = \frac{64 \times 25}{4} = 400$$

$\therefore$ area of bigger triangle = 400 sq.cm.

Ex. (3) In trapezium ABCD, side AB || side CD, diagonal AC and BD intersect each other at point P. Then prove that $\frac{A(\Delta ABP)}{A(\Delta CPD)} = \frac{AB^2}{CD^2}$.

Solution : In trapezium ABCD side AB || side CD

In Δ APB and ΔCPD

$\angle PAB \cong \angle PCD$ ...... alternate angles

$\angle APB \cong \angle CPD$ .... opposite angles

$\therefore \Delta APB \sim \Delta CPD$ .... AA test of similarity

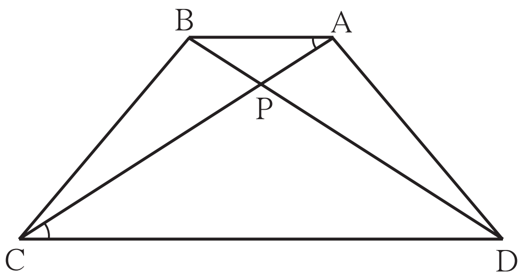


Fig. 1.65

$$\frac{A(\Delta APB)}{A(\Delta CPD)} = \frac{AB^2}{CD^2}$$ . . . . theorem of areas of similar triangles

Practice set 1.4

1. The ratio of corresponding sides of similar triangles is 3 : 5; then find the ratio of their areas .

2. If $\Delta ABC \sim \Delta PQR$ and AB: PQ = 2:3, then fill in the blanks.

$$\frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{AB^2}{\square} = \frac{2^2}{3^2} = \frac{\square}{\square}$$

3. If Δ ABC ~ Δ PQR, A (Δ ABC) = 80, A (Δ PQR) = 125, then fill in the blanks.

$$\frac{A(\Delta ABC)}{A(\Delta \ldots\ldots)} = \frac{80}{125} \qquad \therefore \frac{AB}{PQ} = \frac{\square}{\square}$$

4. Δ LMN ~ Δ PQR, 9 × A (ΔPQR) = 16 × A (ΔLMN). If QR = 20 then find MN.

5. Areas of two similar triangles are 225 sq.cm. 81 sq.cm. If a side of the smaller triangle is 12 cm, then find corresponding side of the bigger triangle .

6. ΔABC and Δ DEF are equilateral triangles. If A(ΔABC) : A (Δ DEF) = 1 : 2 and AB = 4, find DE.

7. In figure 1.66, seg PQ || seg DE, A(Δ PQF) = 20 units, PF = 2 DP, then find A($\square$ DPQE) by completing the following activity.

A(Δ PQF) = 20 units, PF = 2 DP, Let us assume DP = x. $\therefore$ PF = $2x$

DF = DP + $\square$ = $\square$ + $\square$ = $3x$

In Δ FDE and Δ FPQ,

$\angle$ FDE $\cong$ $\angle$ .... corresponding angles

$\angle$ FED $\cong$ $\angle$ .... corresponding angles

$\therefore$ Δ FDE ~ Δ FPQ .... AA test

$$\therefore \frac{A(\Delta FDE)}{A(\Delta FPQ)} = \frac{\square}{\square} = \frac{(3x)^2}{(2x)^2} = \frac{9}{4}$$

$$A(\Delta FDE) = \frac{9}{4} A(\Delta FPQ) = \frac{9}{4} \times \square = \square$$

A($\square$ DPQE) = A(Δ FDE) – A(Δ FPQ)

= $\square$ – $\square$

= $\square$

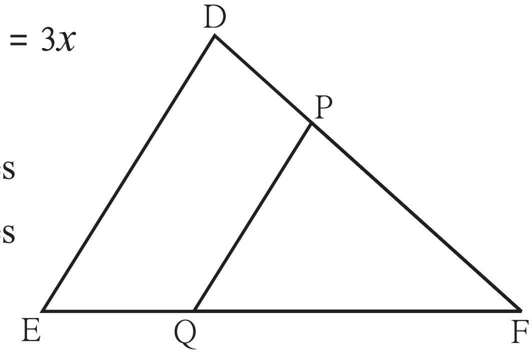


Fig. 1.66

Problem set 1

1. Select the appropriate alternative.

(1) In Δ ABC and Δ PQR, in a one to one correspondence $\frac{AB}{QR} = \frac{BC}{PR} = \frac{CA}{PQ}$ then

(A) Δ PQR ~ Δ ABC

(B) Δ PQR ~ Δ CAB

(C) Δ CBA ~ Δ PQR

(D) Δ BCA ~ Δ PQR

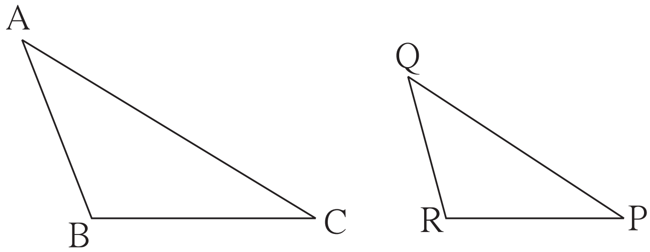


Fig. 1.67

(2) If in Δ DEF and Δ PQR, $\angle D \cong \angle Q$, $\angle R \cong \angle E$ then which of the following statements is false ?

(A) $\frac{EF}{PR} = \frac{DF}{PQ}$ (B) $\frac{DE}{PQ} = \frac{EF}{RP}$

(C) $\frac{DE}{QR} = \frac{DF}{PQ}$ (D) $\frac{EF}{RP} = \frac{DE}{QR}$

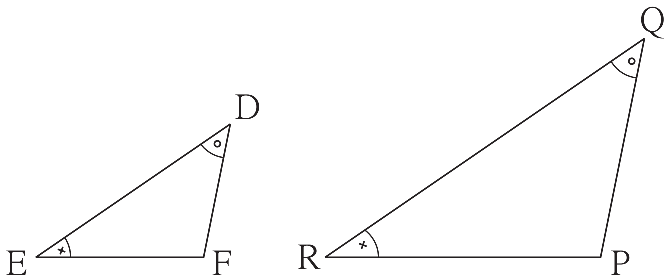


Fig. 1.68

(3) In Δ ABC and Δ DEF $\angle B = \angle E$, $\angle F = \angle C$ and AB = 3DE then which of the statements regarding the two triangles is true ?

(A) The triangles are not congruent and not similar

(B) The triangles are similar but not congruent.

(C) The triangles are congruent and similar.

(D) None of the statements above is true.

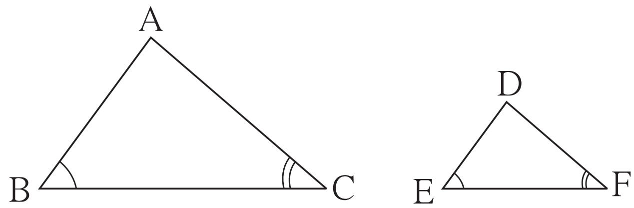


Fig. 1.69

(4) Δ ABC and Δ DEF are equilateral triangles, $A(\Delta ABC):A(\Delta DEF) = 1:2$ If AB = 4 then what is length of DE ?

(A) $2\sqrt{2}$ (B) 4 (C) 8 (D) $4\sqrt{2}$

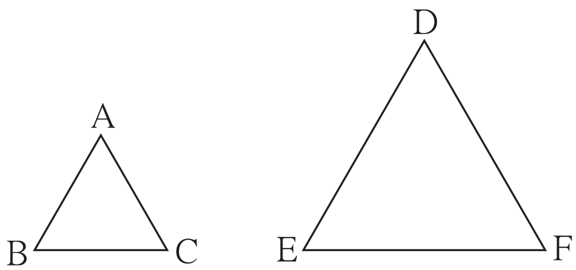


Fig. 1.70

(5) In figure 1.71, seg XY || seg BC, then which of the following statements is true?

(A) $\frac{AB}{AC} = \frac{AX}{AY}$ (B) $\frac{AX}{XB} = \frac{AY}{AC}$

(C) $\frac{AX}{YC} = \frac{AY}{XB}$ (D) $\frac{AB}{YC} = \frac{AC}{XB}$

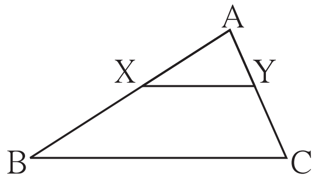


Fig. 1.71

2. In Δ ABC, B – D – C and BD = 7,

BC = 20 then find following ratios.

(1) $\frac{A(\Delta ABD)}{A(\Delta ADC)}$

(2) $\frac{A(\Delta ABD)}{A(\Delta ABC)}$

(3) $\frac{A(\Delta ADC)}{A(\Delta ABC)}$

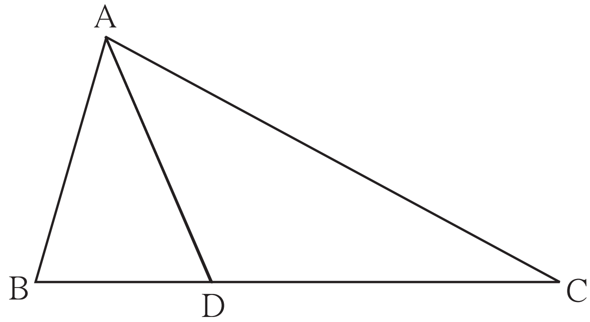


Fig. 1.72

3. Ratio of areas of two triangles with equal heights is 2 : 3. If base of the smaller triangle is 6 cm then what is the corresponding base of the bigger triangle ?

4.

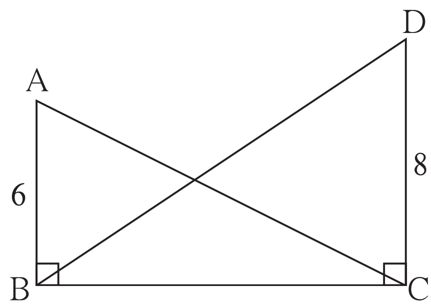


Fig. 1.73

In figure 1.73, $\angle ABC = \angle DCB = 90°$

AB = 6, DC = 8

then $\frac{A(\Delta ABC)}{A(\Delta DCB)} = ?$

5. In figure 1.74, PM = 10 cm

A(ΔPQS) = 100 sq.cm

A(ΔQRS) = 110 sq.cm

then find NR.

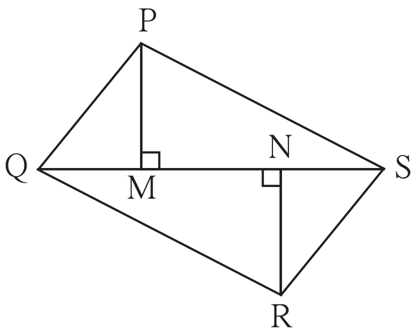


Fig. 1.74

6. Δ MNT ~ Δ QRS. Length of altitude drawn from point T is 5 and length of altitude drawn from point S is 9. Find the ratio $\frac{A(\Delta MNT)}{A(\Delta QRS)}$.

7. In figure 1.75, A−D−C and B−E−C seg DE || side AB If AD = 5, DC = 3, BC = 6.4 then find BE.

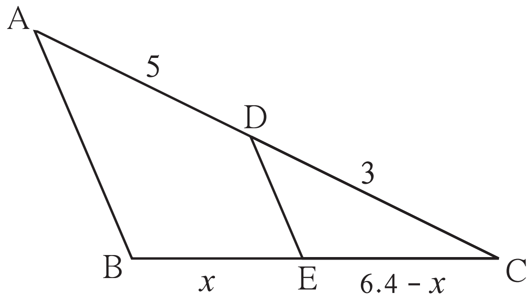


Fig. 1.75

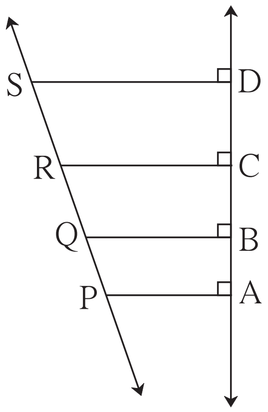


Fig. 1.76

8. In the figure 1.76, seg PA, seg QB, seg RC and seg SD are perpendicular to line AD.

AB = 60, BC = 70, CD = 80, PS = 280 then find PQ, QR and RS.

9. In Δ PQR seg PM is a median. Angle bisectors of ∠PMQ and ∠PMR intersect side PQ and side PR in points X and Y respectively. Prove that XY || QR.

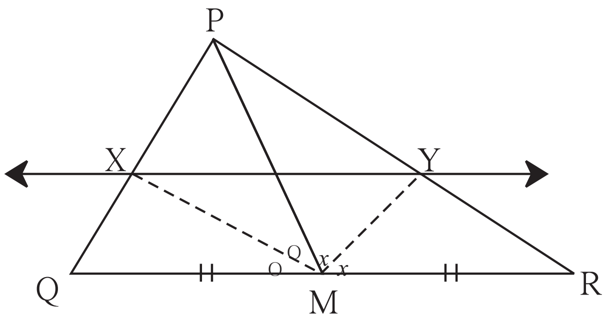


Fig. 1.77

Complete the proof by filling in the boxes.

In Δ PMQ, ray MX is bisector of ∠PMQ.

$\therefore \quad \frac{\square}{\square} = \frac{\square}{\square}$ .... (I) theorem of angle bisector.

In Δ PMR, ray MY is bisector of ∠PMR.

$\therefore \quad \frac{\square}{\square} = \frac{\square}{\square}$ .... (II) theorem of angle bisector.

But $\frac{MP}{MQ} = \frac{MP}{MR}$ .... M is the midpoint QR, hence MQ = MR.

$\therefore \quad \frac{PX}{XQ} = \frac{PY}{YR}$

∴ XY || QR .... converse of basic proportionality theorem.

10. In fig 1.78, bisectors of $\angle$ B and $\angle$ C of Δ ABC intersect each other in point X. Line AX intersects side BC in point Y. AB = 5, AC = 4, BC = 6 then find $\frac{AX}{XY}$.

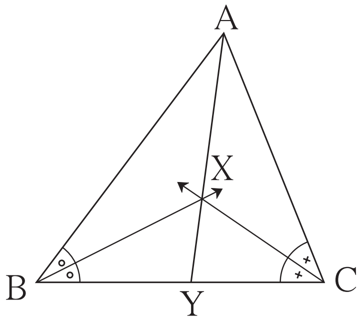


Fig. 1.78

11. In $\square$ ABCD, seg AD || seg BC. Diagonal AC and diagonal BD intersect each other in point P. Then show that $\frac{AP}{PD} = \frac{PC}{BP}$

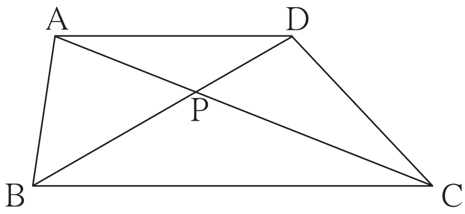


Fig. 1.79

12. In fig 1.80, XY || seg AC. If 2AX = 3BX and XY = 9. Complete the activity to find the value of AC.

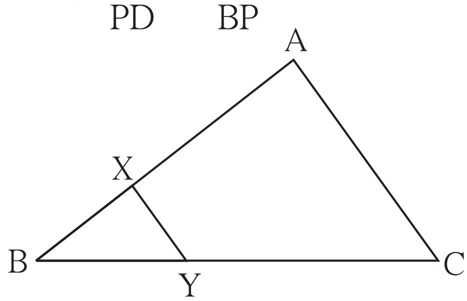


Fig. 1.80

Activity : 2AX = 3BX $\therefore \frac{AX}{BX} = \frac{\square}{\square}$

$\frac{AX + BX}{BX} = \frac{\square + \square}{\square}$ .... by componendo.

$\frac{AB}{BX} = \frac{\square}{\square}$ .... (I)

Δ BCA ~ Δ BYX .... $\square$ test of similarity.

$\therefore \frac{BA}{BX} = \frac{AC}{XY}$ .... corresponding sides of similar triangles.

$\therefore \frac{\square}{\square} = \frac{AC}{9}$ $\therefore$ AC = $\square$ ...from (I)

13★. In figure1.81, the vertices of square DEFG are on the sides of Δ ABC. $\angle$ A = 90°. Then prove that $DE^2 = BD \times EC$

(Hint : Show that Δ GBD is similar to Δ CFE. Use GD = FE = DE.)

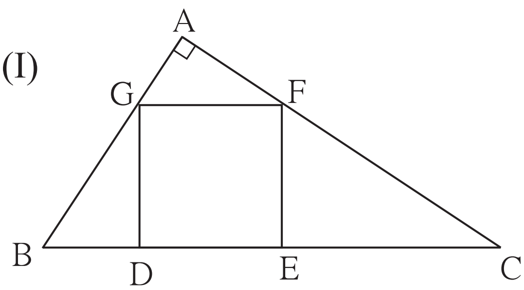


Fig. 1.81

2 Pythagoras Theorem

Let's study.

- Pythagorean triplet
- Theorem of geometric mean
- Application of Pythagoras theorem
- Similarity and right angled triangles
- Pythagoras theorem
- Apollonius theorem

Let's recall.

Pythagoras theorem :

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of remaning two sides.

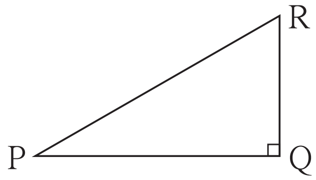


Fig. 2.1

In Δ PQR $\angle$ PQR = 90°

$l(PR)^2 = l(PQ)^2 + l(QR)^2$

We will write this as,

$PR^2 = PQ^2 + QR^2$

The lengths PQ, QR and PR of Δ PQR can also be shown by letters r, p and q. With this convention, refering to figure 2.1, Pythagoras theorem can also be stated as $q^2 = p^2 + r^2$.

Pythagorean Triplet :

In a triplet of natural numbers, if the square of the largest number is equal to the sum of the squares of the remaining two numbers then the triplet is called Pythagorean triplet.

For Example: In the triplet (11, 60, 61) ,

$11^2 = 121$, $60^2 = 3600$, $61^2 = 3721$ and $121 + 3600 = 3721$

The square of the largest number is equal to the sum of the squares of the other two numbers.

$\therefore$ 11, 60, 61 is a Pythagorean triplet.

Verify that (3, 4, 5), (5, 12, 13), (8, 15, 17), (24, 25, 7) are Pythagorean triplets.

Numbers in Pythagorean triplet can be written in any order.

For more information

Formula for Pythagorean triplet:

If a, b, c are natural numbers and $a > b$, then $[(a^2 + b^2),(a^2 - b^2),(2ab)]$ is Pythagorean triplet.

$\because$ $(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$ .... (I)

$(a^2 - b^2) = a^4 - 2a^2b^2 + b^4$ .... (II)

$(2ab)^2 = 4a^2b^2$ .... (III)

$\therefore$ by (I), (II) and (III) , $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$

$\therefore$ $[(a^2 + b^2), (a^2 - b^2), (2ab)]$ is Pythagorean Triplet.

This formula can be used to get various Pythagorean triplets.

For example, if we take $a = 5$ and $b = 3$,

$a^2 + b^2 = 34$, $a^2 - b^2 = 16$, $2ab = 30$.

Check that (34, 16, 30) is a Pythagorean triplet.

Assign different values to a and b and obtain 5 Pythagorean triplet.

Last year we have studied the properties of right angled triangle with the angles $30° - 60° - 90°$ and $45° - 45° - 90°$.

(I) Property of $30°-60°-90°$ triangle.

If acute angles of a right angled triangle are $30°$ and $60°$, then the side opposite $30°$angle is half of the hypotenuse and the side opposite to $60°$ angle is $\frac{\sqrt{3}}{2}$ times the hypotenuse.

See figure 2.2. In Δ LMN, $\angle$ L = 30°, $\angle$ N = 60°, $\angle$ M = 90°

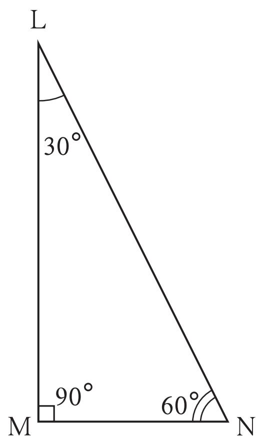


Fig. 2.2

$\therefore$ side opposite 30°angle = MN = $\frac{1}{2} \times$ LN

side opposite 60°angle = LM = $\frac{\sqrt{3}}{2} \times$ LN

If LN = 6 cm, we will find MN and LM.

MN = $\frac{1}{2} \times$ LN	LM = $\frac{\sqrt{3}}{2} \times$ LN
= $\frac{1}{2} \times 6$	= $\frac{\sqrt{3}}{2} \times 6$
= 3 cm	= $3\sqrt{3}$ cm

(II) Property of $45°-45°-90°$

If the acute angles of a right angled triangle are 45° and 45°, then each of the perpendicular sides is $\frac{1}{\sqrt{2}}$ times the hypotenuse.

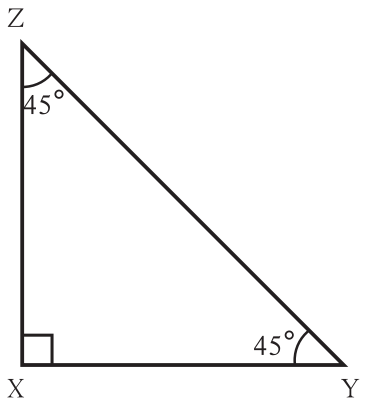


Fig. 2.3

See Figure 2.3. In Δ XYZ,

$$XY = \frac{1}{\sqrt{2}} \times ZY$$

$$XZ = \frac{1}{\sqrt{2}} \times ZY$$

$$\therefore \quad XY = XZ = \frac{1}{\sqrt{2}} \times ZY$$

If $ZY = 3\sqrt{2}$ cm then we will find XY and ZX

$$XY = XZ = \frac{1}{\sqrt{2}} \times 3\sqrt{2}$$

$$XY = XZ = 3\text{cm}$$

In 7th standard we have studied theorem of Pythagoras using areas of four right angled triangles and a square. We can prove the theorem by an alternative method.

Activity:

Take two congruent right angled triangles. Take another isosceles right angled triangle whose congruent sides are equal to the hypotenuse of the two congruent right angled triangles. Join these triangles to form a trapezium

Area of the trapezium = $\frac{1}{2}$ × (sum of the lengths of parallel sides) × height

Using this formula, equating the area of trapezium with the sum of areas of the three right angled triangles we can prove the theorem of Pythagoras.

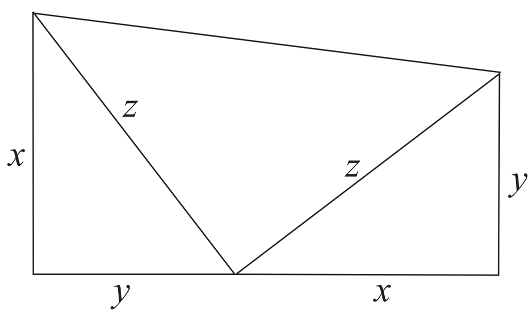


Fig. 2.4

Let's learn.

Now we will give the proof of Pythagoras theorem based on properties of similar triangles. For this, we will study right angled similar triangles.

Similarity and right angled triangle

Theorem : In a right angled triangle, if the altitude is drawn to the hypotenuse, then the two triangles formed are similar to the original triangle and to each other.

Given : In Δ ABC, $\angle$ ABC = 90°,

seg BD $\perp$ seg AC, A-D-C

To prove: Δ ADB ~ Δ ABC

Δ BDC ~ Δ ABC

Δ ADB ~ Δ BDC

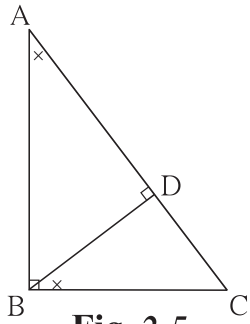


Fig. 2.5

Proof : In Δ ADB and Δ ABC

$\angle$ DAB $\cong$ $\angle$ BAC ...(common angle)

$\angle$ ADB $\cong$ $\angle$ ABC ... (each 90°)

Δ ADB ~ Δ ABC ... (AA test)... (I)

In Δ BDC and Δ ABC

$\angle$ BCD $\cong$ $\angle$ ACB .....(common angle)

$\angle$ BDC $\cong$ $\angle$ ABC ..... (each 90°)

Δ BDC ~ Δ ABC ..... (AA test) ... (II)

$\therefore$ Δ ADB ~ Δ BDC from (I) and (II) ....(III)

$\therefore$ from (I), (II) and (III), Δ ADB ~ Δ BDC ~ Δ ABC ....(transitivity)

Theorem of geometric mean

In a right angled triangle, the perpendicular segment to the hypotenuse from the opposite vertex, is the geometric mean of the segments into which the hypotenuse is divided.

Proof : In right angled triangle PQR, seg QS $\perp$ hypotenuse PR

Δ QSR ~ Δ PSQ .... (similarity of right triangles)

$$\frac{QS}{PS} = \frac{SR}{SQ}$$

$$\frac{QS}{PS} = \frac{SR}{QS}$$

QS^2 = PS × SR

$\therefore$ seg QS is the 'geometric mean' of seg PS and SR.

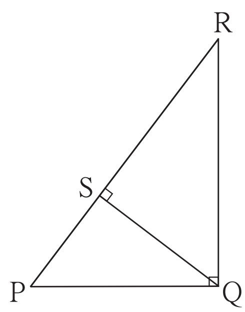


Fig. 2.6

Pythagoras Theorem

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of remaining two sides.

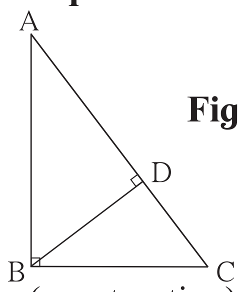


Fig. 2.7

Given : In Δ ABC, $\angle$ABC = 90°

To prove : $AC^2 = AB^2 + BC^2$

Construction : Draw perpendicular seg BD on side AC.

A-D-C.

Proof : In right angled Δ ABC, seg BD $\perp$ hypotenuse AC ..... (construction)

$\therefore \Delta ABC \sim \Delta ADB \sim \Delta BDC$ ..... (similarity of right angled triangles)

$\Delta ABC \sim \Delta ADB$

$\frac{AB}{AD} = \frac{BC}{DB} = \frac{AC}{AB}$ - corresponding sides

$\frac{AB}{AD} = \frac{AC}{AB}$

$AB^2 = AD \times AC$ .... (I)

Similarly, $\Delta ABC \sim \Delta BDC$

$\frac{AB}{BD} = \frac{BC}{DC} = \frac{AC}{BC}$ -corresponding sides

$\frac{BC}{DC} = \frac{AC}{BC}$

$BC^2 = DC \times AC$ .... (II)

Adding (I) and (II)

$AB^2 + BC^2 = AD \times AC + DC \times AC$

$= AC\,(AD + DC)$

$= AC \times AC$ .... (A-D-C)

$\therefore AB^2 + BC^2 = AC^2$

$\therefore AC^2 = AB^2 + BC^2$

Converse of Pythagoras theorem

In a triangle if the square of one side is equal to the sum of the squares of the remaining two sides, then the triangle is a right angled triangle.

Given : In Δ ABC, $AC^2 = AB^2 + BC^2$

To prove : $\angle$ ABC = 90°

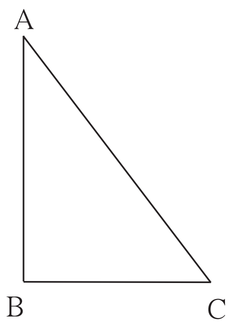


Fig. 2.8

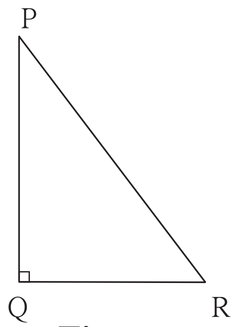


Fig. 2.9

Construction : Draw Δ PQR such that, AB = PQ, BC = QR , $\angle$ PQR = 90°.

Proof : In Δ PQR, $\angle$ Q = 90°

$PR^2 = PQ^2 + QR^2$ .... (Pythagoras theorem)

$= AB^2 + BC^2$ .... (construction) ......(I)

$= AC^2$ .... (given) ......(II)

$\therefore PR^2 = AC^2$

$\therefore$ PR = AC .... (III)

$\therefore \Delta ABC \cong \Delta PQR$ .... (SSS test)

$\therefore \angle ABC = \angle PQR = 90°$

Remember this !

(1) (a) Similarity and right angled triangle

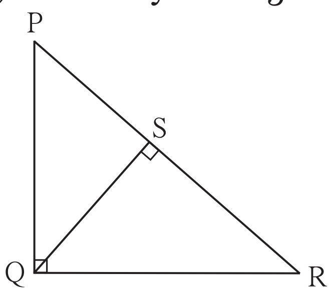


Fig. 2.10

In ΔPQR $\angle$ Q = 90° , seg QS $\perp$ seg PR, $\Delta PQR \sim \Delta PSQ \sim \Delta QSR$. Thus all the right angled triangles in the figure are similar to one another.

(b) Theorem of geometric mean

In the above figure, $\Delta PSQ \sim \Delta QSR$

$\therefore QS^2 = PS \times SR$

$\therefore$ seg QS is the geometric mean of seg PS and seg SR

(2) Pythagoras Theorem:

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of remaining two sides.

(3) Converse of Pythagoras Theorem:

In a triangle, if the square of one side is equal to the sum of the squares of the remaining two sides, then the triangle is a right angled triangle

(4) Let us remember one more very useful property.

In a right angled triangle, if one side is half of the hypotenuse then the angle opposite to that side is 30°.

This property is the converse of 30°-60°-90° theorem.

Ex. (1) See fig 2.11. In Δ ABC, $\angle$B= 90°, $\angle$A= 30°, AC=14, then find AB and BC

Solution :

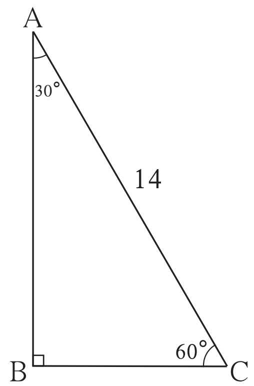


Fig. 2.11

In Δ ABC,

$\angle B = 90°, \angle A = 30°, \therefore \angle C = 60°$

By 30°- 60°- 90° theorem,

$BC = \frac{1}{2} \times AC$

$BC = \frac{1}{2} \times 14$

$BC = 7$

$AB = \frac{\sqrt{3}}{2} \times AC$

$AB = \frac{\sqrt{3}}{2} \times 14$

$AB = 7\sqrt{3}$

Ex. (2) See fig 2.12, In Δ ABC, seg AD $\perp$ seg BC, $\angle$C = 45°, BD = 5 and AC = $8\sqrt{2}$ then find AD and BC.

Solution : In Δ ADC

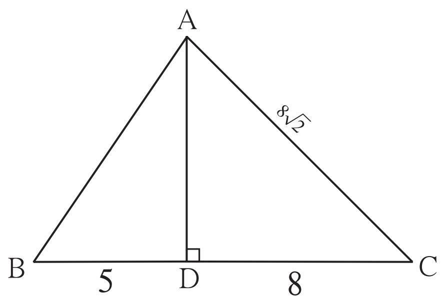


Fig. 2.12

$\angle ADC = 90°, \angle C = 45°, \therefore \angle DAC = 45°$

$AD = DC = \frac{1}{\sqrt{2}} \times 8\sqrt{2}$ ... by 45°-45°-90° theorem

$DC = 8 \quad \therefore AD = 8$

$BC = BD + DC$

$= 5 + 8$

$BC = 13$

Ex. (3) In fig 2.13, $\angle$ PQR = 90°,seg QN $\perp$ seg PR, PN = 9, NR = 16. Find QN.

Solution : In Δ PQR, seg QN $\perp$ seg PR

$NQ^2 = PN \times NR$ ... theorem of geometric mean

$\therefore NQ = \sqrt{PN \times NR}$

$= \sqrt{9 \times 16}$

$= 3 \times 4$

$= 12$

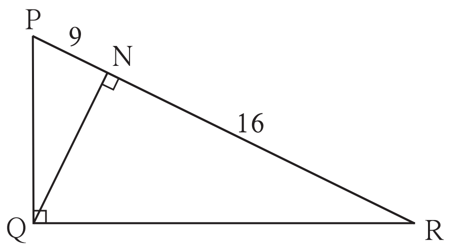


Fig. 2.13

Ex. (4) See figure 2.14.In Δ PQR, $\angle$ PQR = 90°, seg QS $\perp$ seg PR then find x, y, z.

Solution : In Δ PQR, $\angle$ PQR = 90°, seg QS $\perp$ seg PR

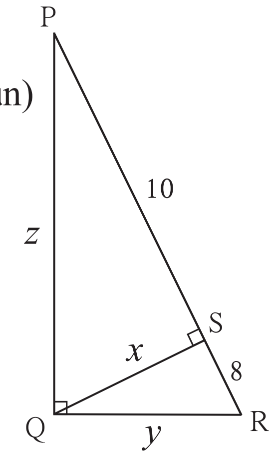


Fig. 2.14

QS = $\sqrt{PS \times SR}$ .... (theorem of geometric mean)

$= \sqrt{10 \times 8}$

$= \sqrt{5 \times 2 \times 8}$

$= \sqrt{5 \times 16}$

$= 4\sqrt{5}$

$\therefore\ x = 4\sqrt{5}$

In Δ QSR, by Pythagoras theorem

$QR^2 = QS^2 + SR^2$

$= \left(4\sqrt{5}\right)^2 + 8^2$

$= 16 \times 5 + 64$

$= 80 + 64$

$= 144$

$\therefore$ QR = 12

In Δ PSQ, by Pythagoras theorem

$PQ^2 = QS^2 + PS^2$

$= \left(4\sqrt{5}\right)^2 + 10^2$

$= 16 \times 5 + 100$

$= 80 + 100$

$= 180$

$= 36 \times 5$

$\therefore$ PQ $= 6\sqrt{5}$

Hence $x = 4\sqrt{5}$, $y = 12$, $z = 6\sqrt{5}$

Ex. (5) In the right angled triangle, sides making right angle are 9 cm and 12 cm. Find the length of the hypotenuse

Solution: In Δ PQR ,$\angle$ Q = 90°

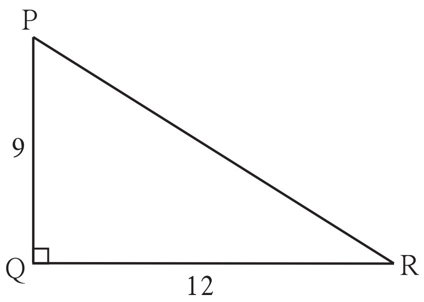


Fig. 2.15

$PR^2 = PQ^2 + QR^2$ (Pythagoras theorem)

$= 9^2 + 12^2$

$= 81 + 144$

$PR^2 = 225$

PR = 15

Hypotenuse= 15 cm

Ex. (6) In Δ LMN , $l = 5$, $m = 13$, $n = 12$. State whether ΔLMN is a right angled triangle or not.

Solution : $l = 5$, $m = 13$, $n = 12$

$l^2 = 25$, $m^2 = 169$, $n^2 = 144$

$\therefore m^2 = l^2 + n^2$

$\therefore$ by converse of Pythagoras theorem Δ LMN is a right angled triangle.

Ex. (7) See fig 2.16. In Δ ABC, seg AD $\perp$ seg BC. Prove that:

$AB^2 + CD^2 = BD^2 + AC^2$

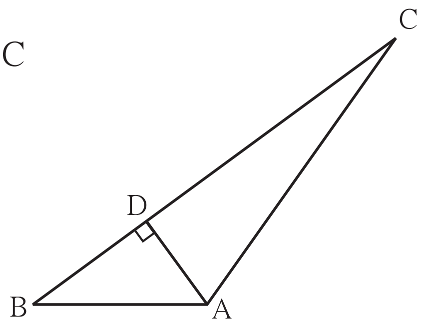


Fig. 2.16

Solution : According to Pythagoras theorem, in Δ ADC

$AC^2 = AD^2 + CD^2$

$\therefore AD^2 = AC^2 - CD^2$ ... (I)

In Δ ADB

$AB^2 = AD^2 + BD^2$

$\therefore AD^2 = AB^2 - BD^2$ ... (II)

$\therefore AB^2 - BD^2 = AC^2 - CD^2$ ....from I and II

$\therefore AB^2 + CD^2 = AC^2 + BD^2$

Practice set 2.1

1. Identify, with reason, which of the following are Pythagorean triplets.

(i)(3, 5, 4) (ii)(4, 9, 12) (iii)(5, 12, 13)

(iv) (24, 70, 74) (v)(10, 24, 27) (vi)(11, 60, 61)

2. In figure 2.17, $\angle$ MNP = 90°, seg NQ $\perp$seg MP, MQ = 9, QP = 4, find NQ.

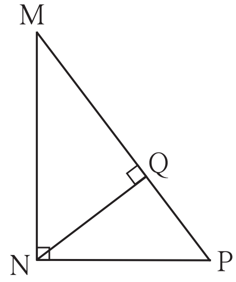


Fig. 2.17

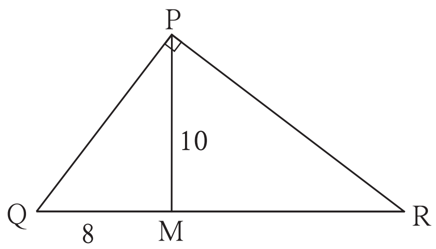


Fig. 2.18

3. In figure 2.18, $\angle$ QPR = 90°, seg PM $\perp$ seg QR and Q-M-R, PM = 10, QM = 8, find QR.

4. See figure 2.19. Find RP and PS using the information given in Δ PSR.

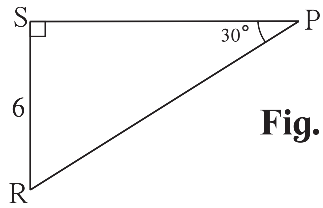


Fig. 2.19

5. For finding AB and BC with the help of information given in figure 2.20, complete following activity.

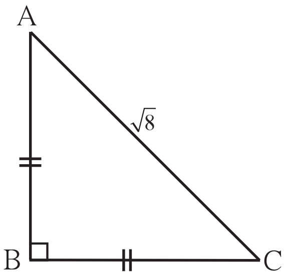


Fig. 2.20

AB = BC .... [ ]

$\therefore \angle BAC =$ [ ]

$\therefore AB = BC =$ [ ] $\times AC$

$=$ [ ] $\times \sqrt{8}$

$=$ [ ] $\times 2\sqrt{2}$

$=$ [ ]

6. Find the side and perimeter of a square whose diagonal is 10 cm.

7. In figure 2.21, $\angle DFE = 90°$, $FG \perp ED$, If GD = 8, FG = 12, find (1) EG (2) FD and (3) EF

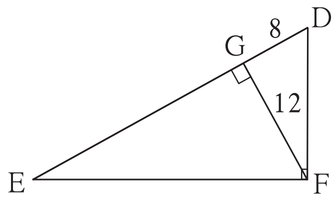


Fig. 2.21

8. Find the diagonal of a rectangle whose length is 35 cm and breadth is 12 cm.

9★. In the figure 2.22, M is the midpoint of QR. $\angle PRQ = 90°$. Prove that, $PQ^2 = 4PM^2 - 3PR^2$

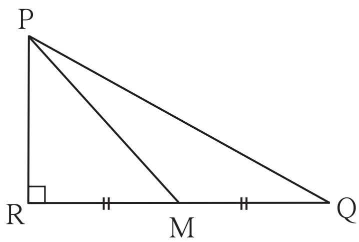


Fig. 2.22

10★. Walls of two buildings on either side of a street are parellel to each other. A ladder 5.8 m long is placed on the street such that its top just reaches the window of a building at the height of 4 m. On turning the ladder over to the other side of the street, its top touches the window of the other building at a height 4.2 m. Find the width of the street.

Let's learn.

Application of Pythagoras theorem

In Pythagoras theorem, the relation between hypotenuse and sides making right angle i.e. the relation between side opposite to right angle and the remaining two sides is given.

In a triangle, relation between the side opposite to acute angle and remaining two sides and relation of the side opposite to obtuse angle with remaining two sides can be determined with the help of Pythagoras theorem. Study these relations from the following examples.

Ex. (1) In Δ ABC, $\angle$ C is an acute angle, seg AD $\perp$ seg BC. Prove that:

$AB^2 = BC^2 + AC^2 - 2BC \times DC$

In the given figure let AB = c, AC = b, AD = p, BC = a, DC = x,

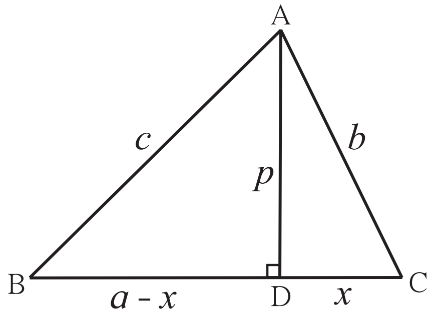


Fig. 2.23

$\therefore$ BD = $a - x$

In Δ ADB , by Pythagoras theorem

$c^2 = (a-x)^2 + \square$

$c^2 = a^2 - 2ax + x^2 + \square$ .... (I)

In Δ ADC, by Pythagoras theorem

$b^2 = p^2 + \square$

$p^2 = b^2 - \square$ .... (II)

Substituting value of p^2 from (II) in (I),

$c^2 = a^2 - 2ax + x^2 + b^2 - x^2$

$\therefore c^2 = a^2 + b^2 - 2ax$

$\therefore AB^2 = BC^2 + AC^2 - 2BC \times DC$

Ex. (2) In Δ ABC, $\angle$ ACB is obtuse angle, seg AD $\perp$ seg BC. Prove that:

$AB^2 = BC^2 + AC^2 + 2BC \times CD$

In the figure seg AD $\perp$ seg BC

Let AD = p, AC = b, AB = c,

BC = a and DC = x.

DB = $a + x$

In Δ ADB, by Pythagoras theorem,

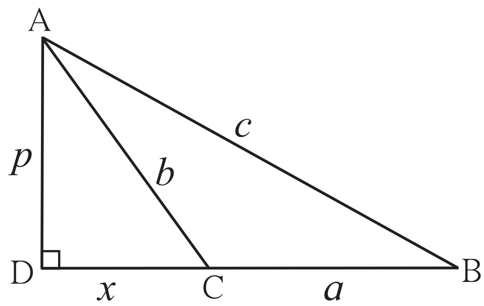


Fig. 2.24

$c^2 = (a + x)^2 + p^2$

$c^2 = a^2 + 2ax + x^2 + p^2$ .... (I)

Similarly, in Δ ADC

$$b^2 = x^2 + p^2$$

$\therefore \quad p^2 = b^2 - x^2$ .... (II)

$\therefore$ substituting the value of p^2 from (II) in (I)

$\therefore \quad c^2 = a^2 + 2ax + b^2$

$\therefore$ $AB^2 = BC^2 + AC^2 + 2BC \times CD$

Apollonius theorem

In Δ ABC, if M is the midpoint of side BC, then $AB^2 + AC^2 = 2AM^2 + 2BM^2$

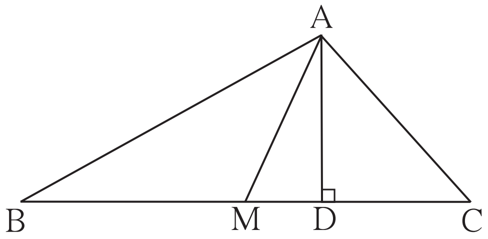


Fig. 2.25

Given : In Δ ABC, M is the midpoint of side BC.

To prove : $AB^2 + AC^2 = 2AM^2 + 2BM^2$

Construction: Draw seg AD $\perp$ seg BC

Proof : If seg AM is not perpendicular to seg BC then out of $\angle$AMB and $\angle$AMC one is obtuse angle and the other is acute angle

In the figure, $\angle$AMB is obtuse angle and $\angle$AMC is acute angle.

From examples (1) and (2) above,

$AB^2 = AM^2 + MB^2 + 2BM \times MD$ ..... (I)

and $AC^2 = AM^2 + MC^2 - 2MC \times MD$

$\therefore$ $AC^2 = AM^2 + MB^2 - 2BM \times MD$ ($\because$ BM = MC) ....(II)

$\therefore$ adding (I) and (II)

$AB^2 + AC^2 = 2AM^2 + 2BM^2$

Write the proof yourself if seg AM $\perp$ seg BC.

From this example we can see the relation among the sides and medians of a triangle.

This is known as Apollonius theorem.

Solved Examples

Ex. (1) In the figure 2.26, seg PM is a median of ΔPQR. PM = 9 and $PQ^2 + PR^2 = 290$, then find QR.

Solution : In Δ PQR , seg PM is a median.

M is the midpoint of seg QR .

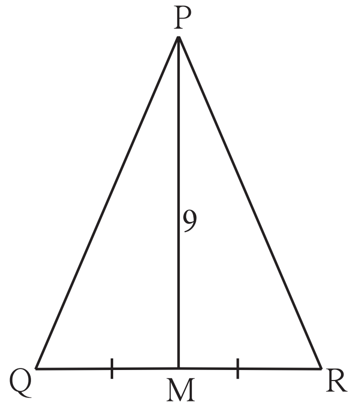


Fig. 2.26

$QM = MR = \frac{1}{2} QR$

$PQ^2 + PR^2 = 2PM^2 + 2QM^2$ (by Apollonius theorem)

$290 = 2 \times 9^2 + 2QM^2$

$290 = 2 \times 81 + 2QM^2$

$290 = 162 + 2QM^2$

$2QM^2 = 290 - 162$

$2QM^2 = 128$

$QM^2 = 64$

$QM = 8$

$\therefore \quad QR = 2 \times QM$

$= 2 \times 8$

$= 16$

Ex. (2) Prove that, the sum of the squares of the diagonals of a rhombus is equal to the sum of the squares of the sides.

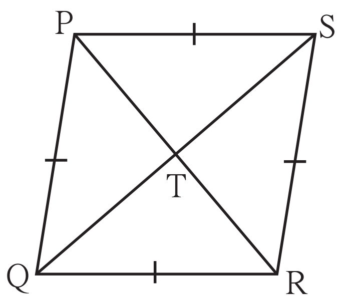


Fig. 2.27

Given : □PQRS is a rhombus. Diagonals PR and SQ intersect each other at point T

To prove : $PS^2 + SR^2 + QR^2 + PQ^2 = PR^2 + QS^2$

Proof : Diagonals of a rhombus bisect each other .

$\therefore$ by Apollonius' theorem,

$PQ^2 + PS^2 = 2PT^2 + 2QT^2$ .... (I)

$QR^2 + SR^2 = 2RT^2 + 2QT^2$ .... (II)

$\therefore$ adding (I) and (II) ,

$PQ^2 + PS^2 + QR^2 + SR^2 = 2(PT^2 + RT^2) + 4QT^2$

$= 2(PT^2 + PT^2) + 4QT^2$ .... (RT = PT)

$= 4PT^2 + 4QT^2$

$= (2PT)^2 + (2QT)^2$

$= PR^2 + QS^2$

(The above proof can be written using Pythagoras theorem also.)

Practice set 2.2

1. In Δ PQR, point S is the midpoint of side QR.If PQ = 11,PR = 17, PS =13, find QR.

2. In Δ ABC, AB = 10, AC = 7, BC = 9 then find the length of the median drawn from point C to side AB

3. In the figure 2.28 seg PS is the median of Δ PQR and PT $\perp$ QR.

Prove that,

(1) $PR^2 = PS^2 + QR \times ST + \left(\frac{QR}{2}\right)^2$

ii) $PQ^2 = PS^2 - QR \times ST + \left(\frac{QR}{2}\right)^2$

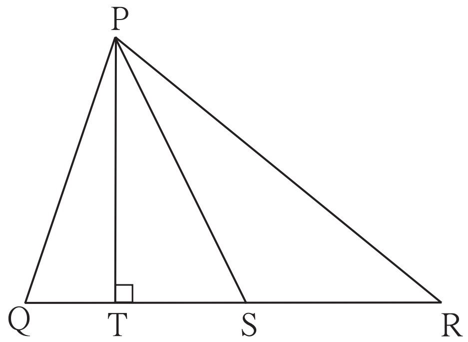


Fig. 2.28

4. In Δ ABC, point M is the midpoint of side BC.

If, $AB^2 + AC^2 = 290\ cm^2$,

AM = 8 cm, find BC.

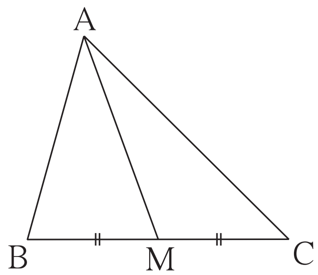


Fig. 2.29

5★. In figure 2.30, point T is in the interior of rectangle PQRS,

Prove that, $TS^2 + TQ^2 = TP^2 + TR^2$

(As shown in the figure, draw seg AB || side SR and A-T-B)

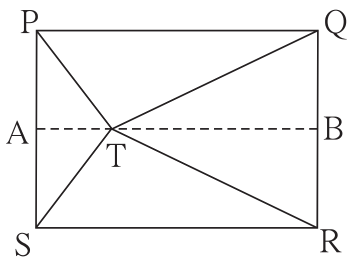


Fig. 2.30

Problem set 2

1. Some questions and their alternative answers are given. Select the correct alternative.

(1) Out of the following which is the Pythagorean triplet?

(A) (1, 5, 10) (B) (3, 4, 5) (C) (2, 2, 2) (D) (5, 5, 2)

(2) In a right angled triangle, if sum of the squares of the sides making right angle is 169 then what is the length of the hypotenuse?

(A) 15 (B) 13 (C) 5 (D) 12

(3) Out of the dates given below which date constitutes a Pythagorean triplet ?

(A) 15/08/17 (B) 16/08/16 (C) 3/5/17 (D) 4/9/15

(4) If a, b, c are sides of a triangle and $a^2 + b^2 = c^2$, name the type of triangle.

(A) Obtuse angled triangle (B) Acute angled triangle

(C) Right angled triangle (D) Equilateral triangle

(5) Find perimeter of a square if its diagonal is $10\sqrt{2}$ cm.

(A)10 cm (B) $40\sqrt{2}$ cm (C) 20 cm (D) 40 cm

(6) Altitude on the hypotenuse of a right angled triangle divides it in two parts of lengths 4 cm and 9 cm. Find the length of the altitude.

(A) 9 cm (B) 4 cm (C) 6 cm (D) $2\sqrt{6}$ cm

(7) Height and base of a right angled triangle are 24 cm and 18 cm find the length of its hypotenus

(A) 24 cm (B) 30 cm (C) 15 cm (D) 18 cm

(8) In Δ ABC, AB = $6\sqrt{3}$ cm, AC = 12 cm, BC = 6 cm. Find measure of $\angle$A.

(A) 30° (B) 60° (C) 90° (D) 45°

2. Solve the following examples.

(1) Find the height of an equilateral triangle having side $2a$.

(2) Do sides 7 cm , 24 cm, 25 cm form a right angled triangle ? Give reason.

(3) Find the length a diagonal of a rectangle having sides 11 cm and 60cm.

(4) Find the length of the hypotenuse of a right angled triangle if remaining sides are 9 cm and 12 cm.

(5) A side of an isosceles right angled triangle is x. Find its hypotenuse.

(6) In Δ PQR; PQ = $\sqrt{8}$, QR = $\sqrt{5}$, PR = $\sqrt{3}$. Is Δ PQR a right angled triangle ? If yes, which angle is of 90° ?

3. In Δ RST, $\angle$ S = 90°, $\angle$ T = 30°, RT = 12 cm then find RS and ST.

4. Find the diagonal of a rectangle whose length is 16 cm and area is 192 sq.cm.

5★. Find the length of the side and perimeter of an equilateral triangle whose height is $\sqrt{3}$ cm.

6. In Δ ABC seg AP is a median. If BC = 18, $AB^2 + AC^2 = 260$ Find AP.

7★. Δ ABC is an equilateral triangle. Point P is on base BC such that PC = $\frac{1}{3}$ BC, if AB = 6 cm find AP.

8. From the information given in the figure 2.31, prove that
PM = PN = $\sqrt{3} \times a$

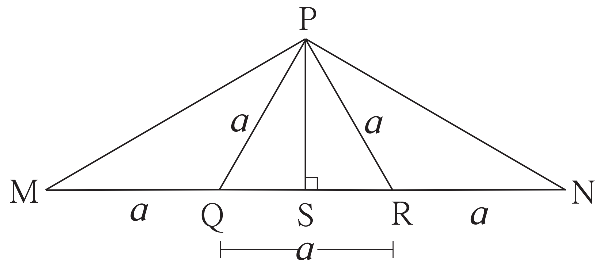


Fig. 2.31

9. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

10. Pranali and Prasad started walking to the East and to the North respectively, from the same point and at the same speed. After 2 hours distance between them was $15\sqrt{2}$ km. Find their speed per hour.

11★. In Δ ABC, $\angle$ BAC = 90°,
seg BL and seg CM are medians of Δ ABC. Then prove that:
$4(BL^2 + CM^2) = 5\ BC^2$

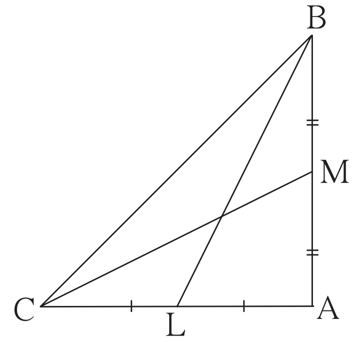


Fig. 2.32

12. Sum of the squares of adjacent sides of a parallelogram is 130 sq.cm and length of one of its diagonals is14 cm. Find the length of the other diagonal.

13. In Δ ABC, seg AD $\perp$ seg BC
DB = 3CD. Prove that :
$2AB^2 = 2AC^2 + BC^2$

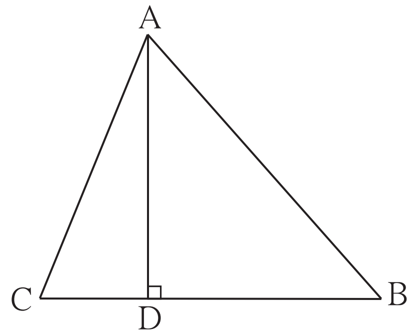


Fig. 2.33

14★. In an isosceles triangle, length of the congruent sides is 13 cm and its base is 10 cm. Find the distance between the vertex opposite the base and the centroid.

15. In a trapezium ABCD,

seg AB || seg DC

seg BD $\perp$ seg AD,

seg AC $\perp$ seg BC,

If AD = 15, BC = 15

and AB = 25. Find A($\square$ ABCD)

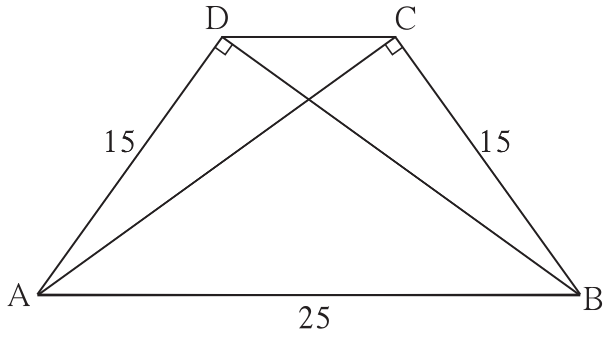


Fig. 2.34

16*. In the figure 2.35, Δ PQR is an equilatral triangle. Point S is on seg QR such that

$QS = \frac{1}{3} QR$.

Prove that : $9 PS^2 = 7 PQ^2$

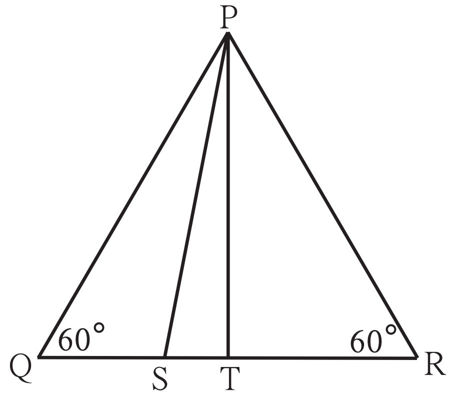


Fig. 2.35

17*. Seg PM is a median of Δ PQR. If PQ = 40, PR = 42 and PM = 29, find QR.

18. Seg AM is a median of Δ ABC. If AB = 22, AC = 34, BC = 24, find AM

ICT Tools or Links

Obtain information on 'the life of Pythagoras' from the internet. Prepare a slide show.

PU5EHH

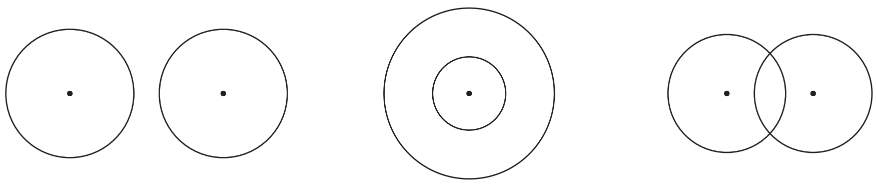
3 Circle

Let's study.

- Circles passing through one, two,three points
- Circles touching each other
- Inscribed angle and intercepted arc
- Secant tangent angle theorem
- Secant and tangent
- Arc of a circle
- Cyclic quadrilateral
- Theorem of intersecting chords

Let's recall.

You are familiar with the concepts regarding circle, like - centre, radius, diameter, chord, interior and exterior of a circle. Also recall the meanings of - congruent circles, concentric circles and intersecting circles.



congruent circles   concentric circles   intersecting circles

Recall the properties of chord studied in previous standard and perform the activity below.

Activity I : In the adjoining figure, seg DE is a chord of a circle with centre C. seg CF ⊥ seg DE. If diameter of the circle is 20 cm, DE =16 cm find CF.

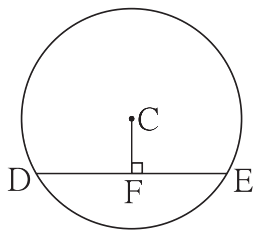


Fig. 3.1

Recall and write theorems and properties which are useful to find the solution of the above problem.

(1) The perpendicular drawn from centre to a chord ____________________

(2) ____________________

(3) ____________________

Using these properties, solve the above problem.

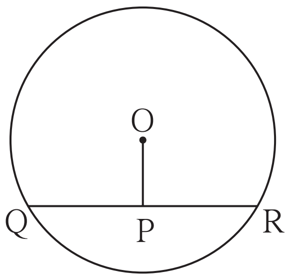


Fig. 3.2

Activity II : In the adjoining figure, seg QR is a chord of the circle with centre O. P is the midpoint of the chord QR. If QR = 24, OP = 10 , find radius of the circle.

To find solution of the problem, write the theorems that are useful.

(1) ____________________

(2) ____________________

Using these theorems solve the problems.

Activity III : In the adjoining figure, M is the centre of the circle and seg AB is a diameter.
seg MS ⊥ chord AD
seg MT ⊥ chord AC
∠DAB ≅ ∠CAB.
Prove that : chord AD ≅ chord AC.

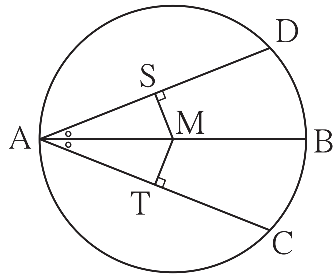


Fig. 3.3

To solve this problem which of the following theorems will you use ?

(1) The chords which are equidistant from the centre are equal in length.

(2) Congruent chords of a circle are equidistant from the centre.

Which of the following tests of congruence of triangles will be useful?

(1) SAS, (2) ASA, (3) SSS, (4) AAS, (5) hypotenuse-side test.

Using appropriate test and theorem write the proof of the above example.

Let's learn.

Circles passing through one, two, three points

In the adjoining figure, point A lies in a plane. All the three circles with centres P, Q, R pass through point A. How many more such circles may pass through point A ?

If your answer is many or innumerable, it is correct.

Infinite number of circles pass through a point.

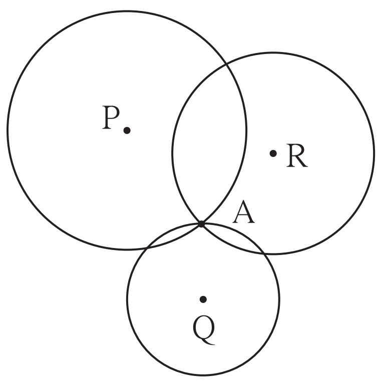


Fig. 3.4

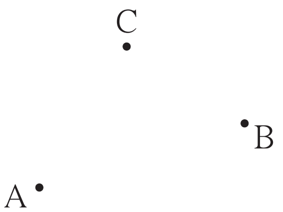


Fig. 3.5

In the adjoining figure, how many circles pass through points A and B?

How many circles contain all the three points A, B, C?

Perform the activity given below and try to find the answer.

ActivityI: Draw segment AB. Draw perpendicular bisector l of the segment AB. Take point P on the line l as centre, PA as radius and draw a circle. Observe that the circle passes through point B also. Find the reason. (Recall the property of perpendicular bisector of a segment.)

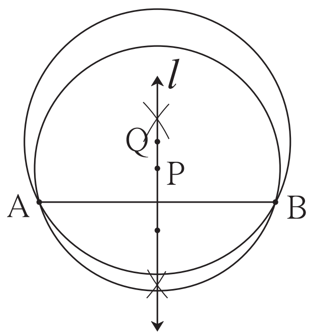


Fig. 3.6

Taking any other point Q on the line l, if a circle is drawn with centre Q and radius QA, will it pass through B ? Think.

How many such circles can be drawn, passing through A and B ? Where will their centres lie?

Activity II : Take any three non-collinear points. What should be done to draw a circle passing through all these points ? Draw a circle passing through these points.
Is it possible to draw one more circle passing through these three points ? Think of it.

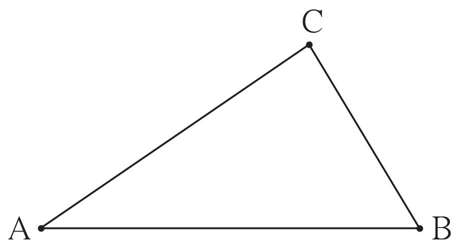


Fig. 3.7

Activity III : Take 3 collinear points D, E, F. Try to draw a circle passing through these points. If you are not able to draw a circle, think of the reason.

Let's recall.

(1) Infinite circles pass through one point.

(2) Infinite circles pass through two distinct points.

(3) There is a unique circle passing through three non-collinear points.

(4) No circle can pass through 3 collinear points.

Let's learn.

Secant and tangent

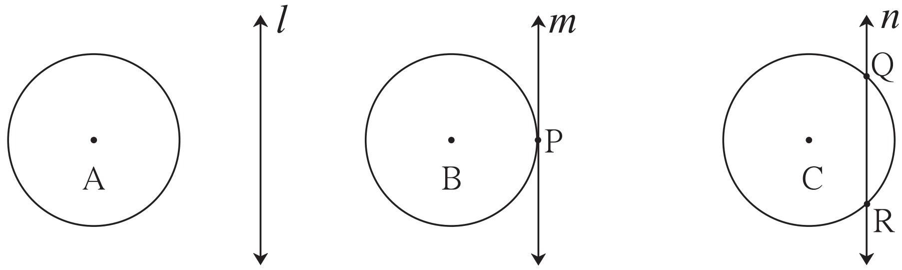


Fig. 3.8

In the figure above, not a single point is common in line l and circle with centre A.

Point P is common to both, line m and circle with centre B. Here, line m is called *a tangent* of the circle and point P is called the point of contact.

Two points Q and R are common to both, the line n and the circle with centre C.

Q and R are intersecting points of line n and the circle. Line n is called *a secant* of the circle .

Let us understand an important property of a tangent from the following activity.

Activity :

Draw a sufficiently large circle with centre O. Draw radius OP. Draw a line AB $\perp$ seg OP. It intersects the circle at points A, B. Imagine the line slides towards point P such that all the time it remains parallel to its original position. Obviously, while the line slides, points A and B approach each other along the circle. At the end, they get merged in point P, but the angle between the radius OP and line AB will remain a right angle.

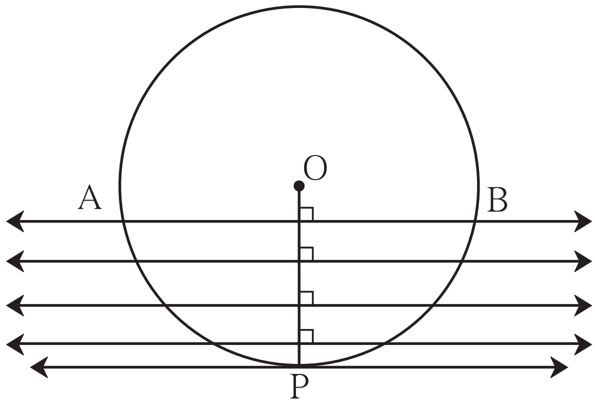


Fig. 3.9

At this stage the line AB becomes a tangent of the circle at P.

So it is clear that, the tangent at any point of a circle is perpendicular to the radius at that point.

This property is known as 'tangent theorem'.

Tangent theorem

Theorem : A tangent at any point of a circle is perpendicular to the radius at the point of contact.

There is an indirect proof of this theorem.

For more information

Given : Line l is a tangent to the circle with centre O at the point of contact A.

To prove : line $l \perp$ radius OA.

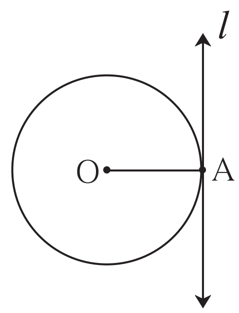


Fig. 3.10

Proof : Assume that, line l is not perpendicular to seg OA.

Suppose, seg OB is drawn perpendicular to line l.

Of course B is not same as A.

Now take a point C on line l such that A-B-C and BA = BC .

Now in, Δ OBC and Δ OBA

seg BC $\cong$ seg BA .... (construction)

$\angle$ OBC $\cong$ $\angle$ OBA .... (each right angle)

seg OB $\cong$ seg OB

$\therefore$ Δ OBC $\cong$ Δ OBA .... (SAS test)

$\therefore$ OC = OA

But seg OA is a radius.

$\therefore$ seg OC must also be radius.

$\therefore$ C lies on the circle.

That means line l intersects the circle in two distinct points A and C.

But line l is a tangent. .... (given)

$\therefore$ it intersects the circle in only one point.

Our assumption that line l is not perpendicular to radius OA is wrong.

$\therefore$ line $l \perp$ radius OA.

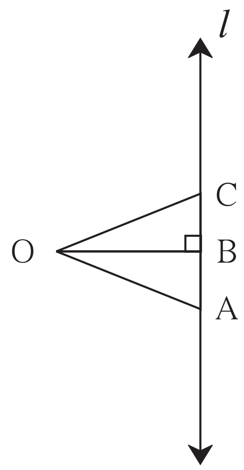


Fig. 3.11

Let's recall.

Which theorems do we use in proving that hypotenuse is the longest side of a right angled triangle?

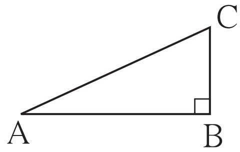


Fig. 3.12

Let's learn.

Converse of tangent theorem

Theorem: A line perpendicular to a radius at its point on the circle is a tangent to the circle.

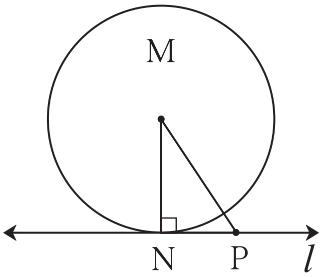


Fig. 3.13

Given : M is the centre of a circle seg MN is a radius.
Line $l \perp$ seg MN at N.

To prove : Line l is a tangent to the circle.

Proof : Take any point P, other that N, on the line l. Draw seg MP.

Now in Δ MNP, $\angle$ N is a right angle.

$\therefore$ seg MP is the hypotenuse.

$\therefore$ seg MP > seg MN.

As seg MN is radius, point P can't be on the circle.

$\therefore$ no other point, except point N, of line l is on the circle.

$\therefore$ line l intersects the circle in only one point N.

$\therefore$ line l is a tangent to the circle.

Let's discuss.

In figure 3.14 , B is a point on the circle with centre A. The tangent of the circle passing through B is to be drawn. There are infinite lines passing through the point B. Which of them will be the tangent ? Can the number of tangents through B be more than one ?

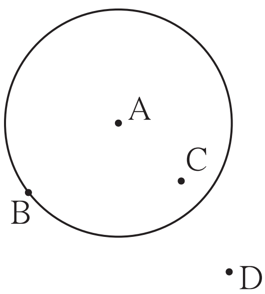


Fig. 3.14

Point C lies in the interior of the circle. Can you draw tangents to the circle through C ?

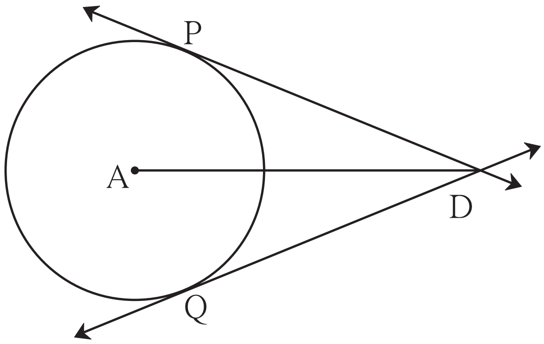


Fig. 3.15

Point D is in the exterior of the circle. Can you draw a tangent to the circle through D? If yes, how many such tangents are possible? From the discussion you must have understood that two tangents can be drawn to a circle from the point outside the circle as shown in the figure.

In the adjoing figure line DP and line DQ. touch the circle at points P and Q. Seg DP and seg DQ are called tangent segments.

Tangent segment theorem

Theorem : Tangent segments drawn from an external point to a circle are congruent.

Observe the adjoining figure. Write 'given' and 'to prove.'

Draw radius AP and radius AQ and complete the following proof of the theorem.

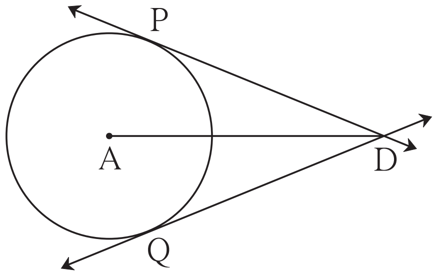


Fig. 3.16

Proof : In Δ PAD and Δ QAD,

seg PA $\cong$ ________ radii of the same circle.

seg AD $\cong$ seg AD ________________

$\angle APD = \angle AQD = 90°$ ...... tangent theorem

$\therefore$ Δ PAD $\cong$ Δ QAD ______________

$\therefore$ seg DP $\cong$ seg DQ ______________

Solved Examples

Ex. (1) In the adjoining figure circle with centre D touches the sides of $\angle$ ACB at A and B. If $\angle$ ACB = 52°, find measure of $\angle$ ADB.

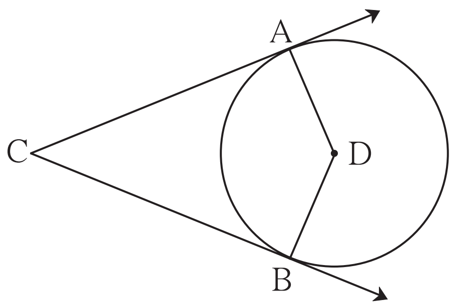


Fig. 3.17

Solution : The sum of all angles of a quadrilateral is 360°.

$\therefore \angle ACB + \angle CAD + \angle CBD + \angle ADB = 360°$

$\therefore 52° + 90° + 90° + \angle ADB = 360°$ .... Tangent theorem

$\therefore \angle ADB + 232° = 360°$

$\therefore \angle ADB = 360° - 232° = 128°$

Eg. (2) Point O is the centre of a circle. Line *a* and line *b* are parallel tangents to the circle at P and Q. Prove that segment PQ is a diameter of the circle.

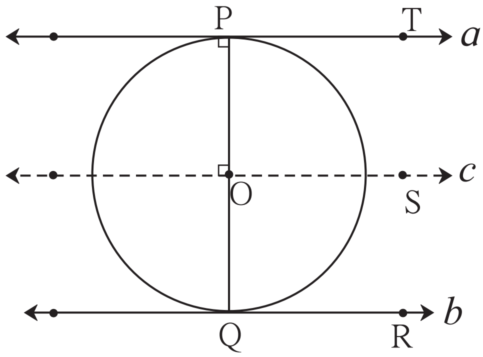


Fig. 3.18

Solution : Draw a line *c* through O which is parallel to line *a*. Draw radii OQ and OP.

Now, $\angle$ OPT = 90° ..... Tangent theorem

$\therefore$ $\angle$SOP = 90°... Int. angle property ... (I)

line *a* || line *c* ..... construction

line *a* || line b ..... given

$\therefore$ line b || line c

$\therefore$ $\angle$SOQ = 90°... Int. angle property ...(II)

$\therefore$ From (I) and (II),

$\angle$ SOP + $\angle$ SOQ = 90° + 90° = 180°

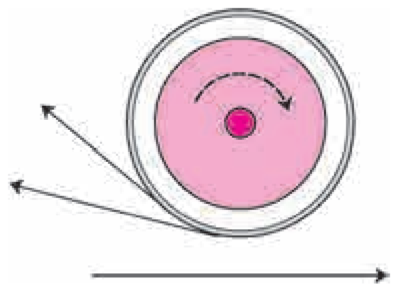
$\therefore$ ray OP and ray OQ are opposite rays.

$\therefore$ P, O, Q are collinear points.

$\therefore$ seg PQ is a diameter of the circle.

When a motor cycle runs on a wet road in rainy season, you may have seen water splashing from its wheels. Those splashes are like tangents of the circle of the wheel. Find out the reason from your science teacher.

Observe the splinters escaping from a splintering wheel in Diwali fire works and while sharpning a knife. Do they also look like tangents ?



Direction of running wheel

Remember this !

(1) Tangent theorem : The tangent at any point of a circle is perpendicular to the radius through the point of contact.

(2) A line perpendicular to a radius at its point on the circle, is a tangent to the circle.

(3) Tangent segments drawn from an external point to a circle are congruent.

Practice set 3.1

1. In the adjoining figure the radius of a circle with centre C is 6 cm, line AB is a tangent at A. Answer the following questions.

(1) What is the measure of $\angle$CAB ? Why ?

(2) What is the distance of point C from line AB? Why ?

(3) d(A,B)= 6 cm, find d(B,C).

(4) What is the measure of $\angle$ABC ? Why ?

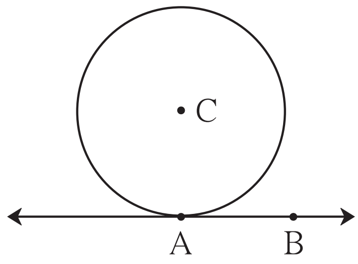


Fig. 3.19

2. In the adjoining figure, O is the centre of the circle. From point R, seg RM and seg RN are tangent segments touching the circle at M and N. If (OR) = 10 cm and radius of the circle = 5 cm, then

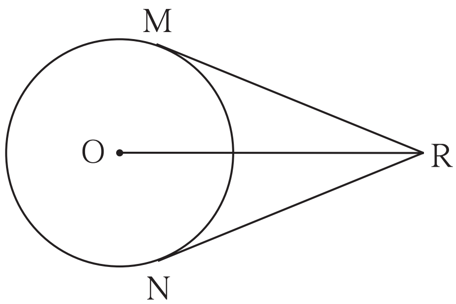


Fig. 3.20

(1) What is the length of each tangent segment ?

(2) What is the measure of $\angle$MRO ? (3) What is the measure of $\angle$ MRN ?

3. Seg RM and seg RN are tangent segments of a circle with centre O. Prove that seg OR bisects $\angle$MRN as well as $\angle$MON.

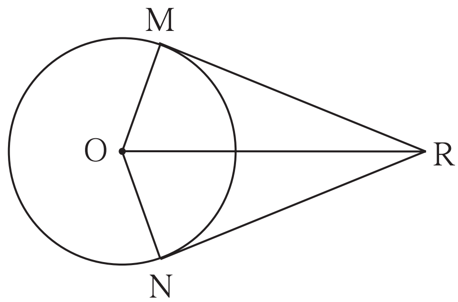


Fig. 3.21

4. What is the distance between two parallel tangents of a circle having radius 4.5 cm ? Justify your answer.

ICT Tools or Links

With the help of Geogebra software, draw a circle and its tangents from a point in its exterior. Check that the tangent segments are congruent.

Let's learn.

Touching circles

Activity I :

Take three collinear points X-Y-Z as shown in figure 3.22. Draw a circle with centre X and radius XY.

Draw another circle with centre Z and radius YZ.

Note that both the circles intersect each other at the single point Y.

Draw a line through point Y and perpendicular to seg XZ.

Note that this line is a common tangent of the two circles.

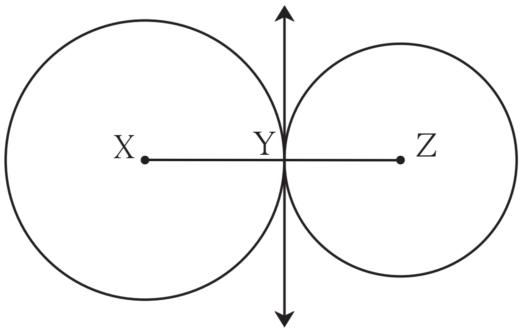


Fig. 3.22

Activity II :

Take points Y-X-Z as shown in the figure 3.23.

Draw a circle with centre Z and radius ZY.

Also draw a circle with centre X and radius XY.

Note that both the circles intersect each other at the point Y.

Draw a line perpendicular to seg YZ through point Y, that is the common tangent for the circles.

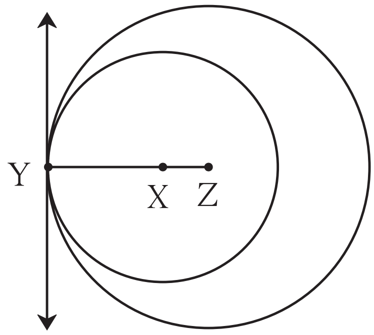


Fig. 3.23

You must have understood, the circles in both the figures are coplaner and intersect at one point only. Such circles are said to be circles touching each other.

Touching circles can be defined as follows.

If two circles in the same plane intersect with a line in the plain in only one point, they are said to be touching circles and the line is their common tangent. The point common to the circles and the line is called their common point of contact.

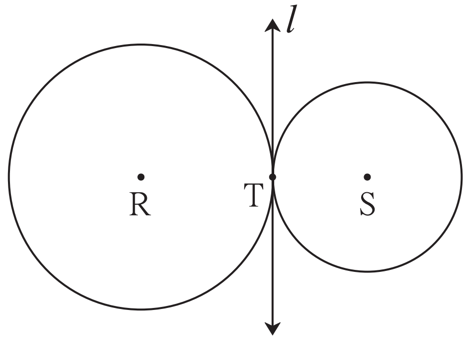


Fig. 3.24

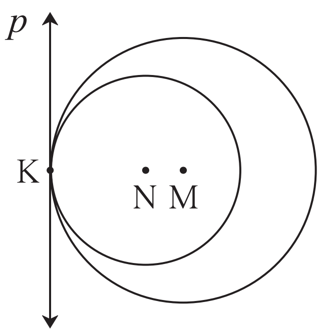


Fig. 3.25

In figure 3.24, the circles with centres R and S touch the line l in point T. So they are two touching circles with l as common tangent. They are touching externally.

In figure 3.25 the circles with centres M, N touch each other internally and line p is their common tangent.

Let's think.

(1) The circles shown in figure 3.24 are called externally touching circles. why ?

(2) The circles shown in figure 3.25 are called internally touching circles. why ?

(3) In figure 3.26, the radii of the circles with centers A and B are 3 cm and 4 cm respectively. Find -

(i) d(A,B) in figure 3.26 (a) (ii) d(A,B) in figure 3.26 (b)

Theorem of touching circles

Theorem : If two circles touch each other, their point of contact lies on the line joining their centres.

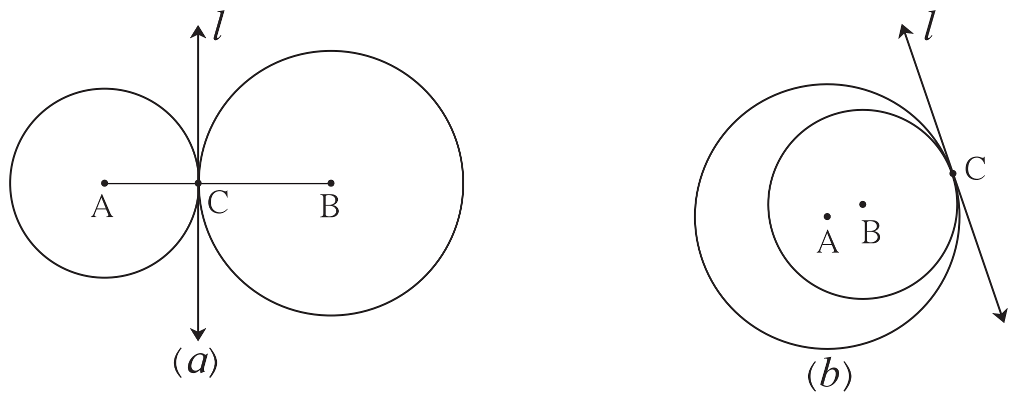


Fig. 3.26

Given : C is the point of contact of the two circles with centers A, B.

To prove : Point C lies on the line AB.

Proof : Let line l be the common tangent passing through C, of the two touching circles. line $l \perp$ seg AC, line $l \perp$ seg BC. $\therefore$ seg AC $\perp$ line l and seg BC $\perp$ line l. Through C, only one line perpendicular to line l can be drawn. $\therefore$ points C, A, B are collinear.

Remember this!

(1) The point of contact of the touching circles lies on the line joining their centres.

(2) If the circles touch each other externally, distance between their centres is equal to the sum of their radii.

(3) The distance between the centres of the circles touching internally is equal to the difference of their radii.

Practice set 3.2

1. Two circles having radii 3.5 cm and 4.8 cm touch each other internally. Find the distance between their centres.

2. Two circles of radii 5.5 cm and 4.2 cm touch each other externally. Find the distance between their centres.

3. If radii of two circles are 4 cm and 2.8 cm. Draw figure of these circles touching each other - (i) externally (ii) internally.

4. In fig 3.27, the circles with centres P and Q touch each other at R. A line passing through R meets the circles at A and B respectively. Prove that -

(1) seg AP || seg BQ,

(2) Δ APR $\sim$ Δ RQB, and

(3) Find $\angle$ RQB if $\angle$ PAR = 35°

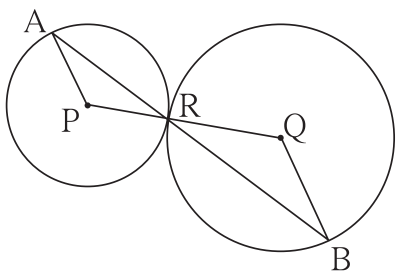


Fig. 3.27

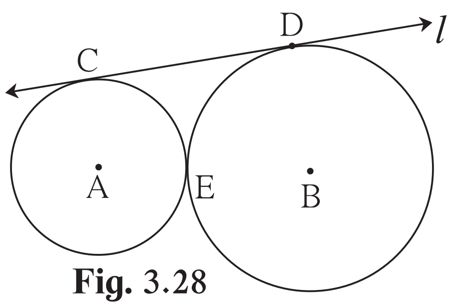


Fig. 3.28

5*. In fig 3.28 the circles with centres A and B touch each other at E. Line l is a common tangent which touches the circles at C and D respectively. Find the length of seg CD if the radii of the circles are 4 cm, 6 cm.

Let's recall.

Arc of a circle

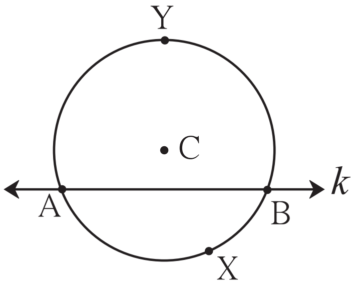


Fig. 3.29

A secant divides a circle in two parts. Any one of these two parts and the common points of the circle and the secant constitute an **arc of the circle.**

The points of intersection of circle and secant are called the end points of the arcs.

In figure 3.29, due to secant k we get two arcs of the circle with centre C– arc AYB, arc AXB.

If the centre of a circle is on one side of the secant then the arc on the side of the centre is called **'major arc'** and the arc which is on the other side of the centre is called **'minor arc'**. In the figure 3.29 arc AYB is a major arc and arc AXB is a minor arc. If there is no confusion then the name of a minor arc is written using its end points only. For example, the arc AXB in figure 3.29, is written as arc AB.

Here after, we are going to use the same convention for writing the names of arcs.

Central angle

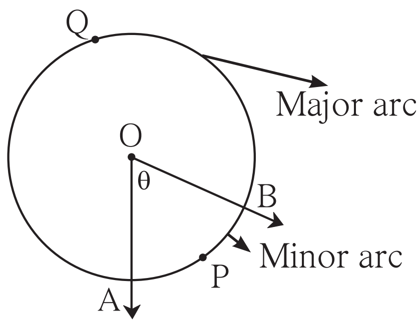


Fig. 3.30

When the vertex of an angle is the centre of a circle, it is called a central angle. In the figure 3.30, O is the centre of a circle and $\angle$AOB is a central angle.

Like secant, a central angle also divides a circle into two arcs.

Measure of an arc

To compare two arcs, we need to know their measures. Measure of an arc is defined as follows.

(1) Measure of a minor arc is equal to the measure of its corresponding central angle. In figure 3.30 measure of central $\angle$ AOB is θ.

$\therefore$ measure of minor arc APB is also θ.

(2) Measure of major arc = 360° – measure of corresponding minor arc.

In figure 3.30 measure of major arc AQB = 360° – measure of minor arc APB

= $360^\circ - \theta$

(3) Measure of a semi circular arc, that is of a semi circle is 180°.

(4) Measure of a complete circle is 360°.

Let's learn.

Congruence of arcs

When two coplanar figures coincide with each other, they are called congruent figures. We know that two angles of equal measure are congruent.

Similarily, are two arcs of the same measure congruent ?

Find the answer of the question by doing the following activity.

Activity :

Draw two circles with centre C, as shown in the figure. Draw $\angle$ DCE, $\angle$ FCG of the same measure and $\angle$ ICJ of different measure.

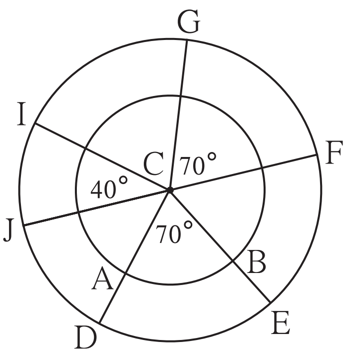


Fig. 3.31

Arms of $\angle$ DCE intersect inner circle at A and B.

Do you notice that the measures of arcs AB and DE are the same ? Do they coincide ? No, definitely not.

Now cut and seperate the sectors C–DE; C–FG and C–IJ. Check whether the arc DE, arc FG and arc IJ coincide with each other.

Did you notice that equality of measures of two arcs is not enough to make the two arcs congruent ? Which additional condition do you think is neccessary to make the two arcs congruent ?

From the above activity -

Two arcs are congruent if their measures and radii are equal.

'Arc DE and arc GF are congruent' is written in symbol as arc DE $\cong$ arc GF.

Property of sum of measures of arcs

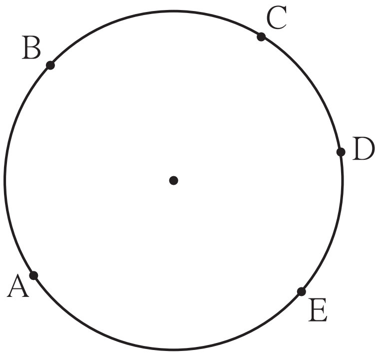


Fig. 3.32

In figure 3.32, the points A, B, C, D, E are concyclic. With these points many arcs are formed. There is one and only one common point C to arc ABC and arc CDE. So measure of arc ACE is the sum of measures of arc ABC and arc CDE.

$m(\text{arc ABC}) + m(\text{arc CDE}) = m(\text{arc ACE})$

But arc ABC and arc BCE have many points in common. [All points on arc BC.]

So $m(\text{arc ABE}) \neq m(\text{arc ABC}) + m(\text{arc BCE})$.

Theorem : The chords corresponding to congruent arcs of a circle (or congruent circles) are congruent.

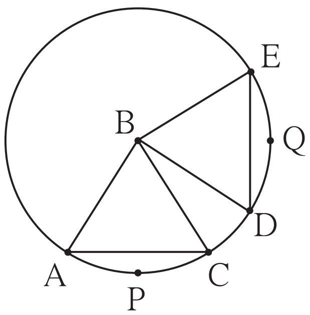


Fig. 3.33

Given : In a circle with centre B arc APC ≅ arc DQE

To Prove : Chord AC ≅ chord DE

Proof : (Fill in the blanks and complete the proof.)

In Δ ABC and Δ DBE,

side AB ≅ side DB (........)

side .... ≅ side .... (........)

∠ABC ≅ ∠DBE measures of congruent arcs

∴ Δ ABC ≅ Δ DBE (.......)

∴ chord AC ≅ chord DE (......)

Theorem : Corresponding arcs of congruent chords of a circle (or congruent circles) are congruent.

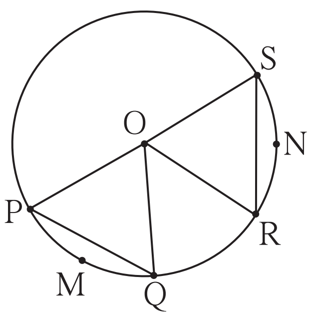


Fig. 3.34

Given : O is the centre of a circle chord PQ ≅ chord RS.

To prove : Arc PMQ ≅ arc RNS

Proof : Consider the following statements and write the proof.

Two arcs are congruent if their measures and radii are equal. Arc PMQ and arc RNS are arcs of the same circle, hence have equal radii.

Their measures are same as the measures of their central angles. To obtain central angles we have to draw radii OP, OQ, OR, OS.

Can you show that Δ OPQ and Δ ORS are congruent ?

Prove the above two theorems for congruent circles.

Let's think.

- While proving the first theorem of the two, we assume that the minor arc APC and minor arc DQE are congruent. Can you prove the same theorem by assuming that corresponding major arcs congruent ?
- In the second theorem, are the major arcs corresponding to congruent chords congruent ? Is the theorem true, when the chord PQ and chord RS are diameters of the circle ?

Solved Examples

Ex. (1) A, B, C are any points on the circle with centre O.

(i) Write the names of all arcs formed due to these points.

(ii) If m arc (BC) = 110° and m arc (AB) = 125°, find measures of all remaining arcs.

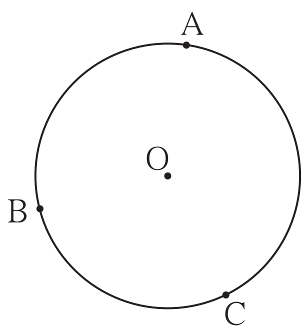


Fig. 3.35

Solution : (i) Names of arcs -

arc AB, arc BC, arc AC, arc ABC, arc ACB, arc BAC

(ii) m(arc ABC) = m(arc AB) + m(arc BC)

$= 125° + 110° = 235°$

m (arc AC) = 360° − m (arc ACB)

$= 360° - 235° = 125°$

Similarly, m(arc ACB) = 360° − 125° = 235°

and m(arc BAC) = 360° − 110° = 250°

Ex. (2) In the figure 3.36 a rectangle PQRS is inscribed in a circle with centre T. Prove that, (i) arc PQ ≅ arc SR

(ii) arc SPQ ≅ arc PQR

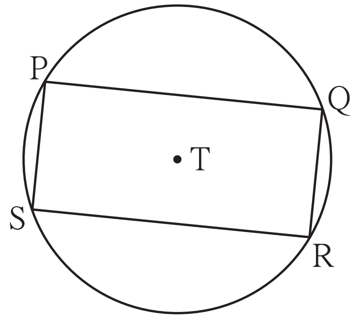


Fig. 3.36

Solution : (i) □ PQRS in a rectangle.

∴ chord PQ ≅ chord SR .... opposite sides of a rectangle

∴ arc PQ ≅ arc SR ...... arcs corresponding to congruent chords.

(ii) chord PS ≅ chord QR ..... Opposite sides of a rectangle

∴ arc SP ≅ arc QR ..... arcs corresponding to congruent chords.

∴ measures of arcs SP and QR are equal

Now, m(arc SP) + m(arc PQ) = m(arc PQ) + m(arc QR)

∴ m(arc SPQ) = m(arc PQR)

∴ arc SPQ ≅ arc PQR

Remember this !

(1) An angle whose vertex is the centre of a circle is called a central angle.

(2) Definition of measure of an arc – (i) The measure of a minor arc is the measure of its central angle. (ii) Measure of a major arc = 360° – measure of its corresponding minor arc. (iii) measure of a semicircle is 180°.

(3) When two arcs are of the same radius and same measure, they are congruent.

(4) When only one point C is common to arc ABC, and arc CDE of the same circle, m(arc ABC) + m(arc CDE) = m(arc ACE)

(5) Chords of the same or congruent circles are equal if the related arcs are congruent.

(6) Arcs of the same or congruent circles are equal if the related chords are congruent.

Practice set 3.3

1. In figure 3.37, points G, D, E, F are concyclic points of a circle with centre C.

∠ ECF = 70°, m(arc DGF) = 200°

find m(arc DE) and m(arc DEF).

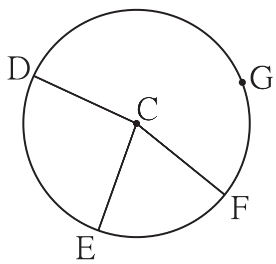


Fig. 3.37

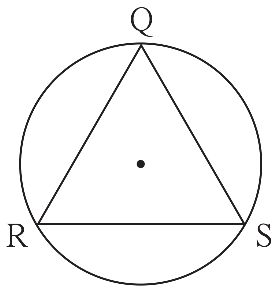


Fig. 3.38

2*. In fig 3.38 Δ QRS is an equilateral triangle. Prove that,

(1) arc RS $\cong$ arc QS $\cong$ arc QR

(2) m(arc QRS) = 240°.

3. In fig 3.39 chord AB $\cong$ chord CD, Prove that,

arc AC $\cong$ arc BD

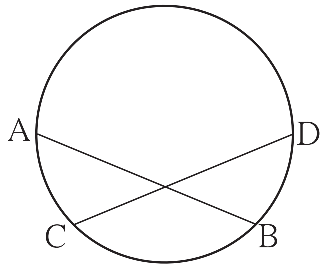


Fig. 3.39

Let's learn.

We have learnt some properties relating to a circle and points as well as lines (tangents). Now let us learn some properties regarding circle and angles with the help of some activities.

Activity I :

Draw a sufficiently large circle of any radius as shown in the figure 3.40. Draw a chord AB and central $\angle$ ACB. Take any point D on the major arc and point E on the minor arc.

(1) Measure $\angle$ADB and $\angle$ACB and compare the measures.

(2) Measure $\angle$ADB and $\angle$AEB. Add the measures.

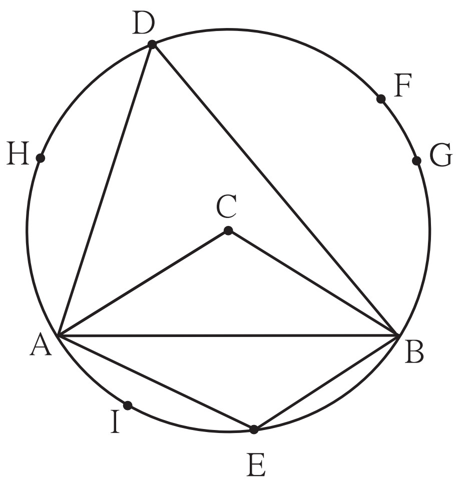


Fig. 3.40

(3) Take points F, G, H on the arc ADB. Measure ∠AFB, ∠AGB, ∠AHB. Compare these measures with each other as well as with measure of ∠ADB.

(4) Take any point I on the arc AEB. Measure ∠AIB and compare it with ∠AEB.

From the activity you must have noticed-

(1) The measure ∠ ACB is twice the measure of ∠ ADB.

(2) The sum of the measures of ∠ ADB and ∠ AEB is $180°$.

(3) The angles ∠ AHB, ∠ ADB, ∠ AFB and ∠ AGB are of equal measure.

(4) The measure of ∠ AEB and ∠ AIB are equal.

Activity II :

Draw a sufficiently large circle with centre C as shown in the figure 3.41. Draw any diameter PQ. Now take points R,S,T on both the semicircles. Measure ∠PRQ, ∠ PSQ, ∠PTQ. Note that each is a right angle.

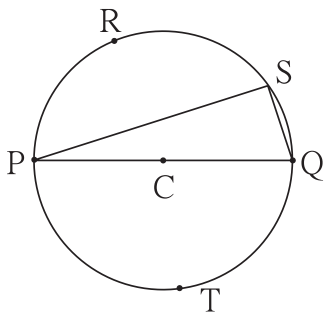


Fig. 3.41

The properties you saw in the above activities are theorems that give relations between circle and angles.

Let us learn some definitions required to prove the theorems.

Inscribed angle

In figure 3.42, C is the centre of a circle. The vertex D, of ∠PDQ lies on the circle. The arms of ∠PDQ intersect the circle at A and B. Such an angle is called an angle inscribed in the circle or in the arc.

In figure 3.42, ∠ADB is inscribed in the arc ADB.

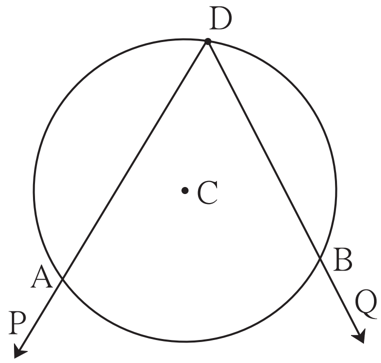


Fig. 3.42

Intercepted arc

Observe all figures (i) to (vi) in the following figure 3.43.

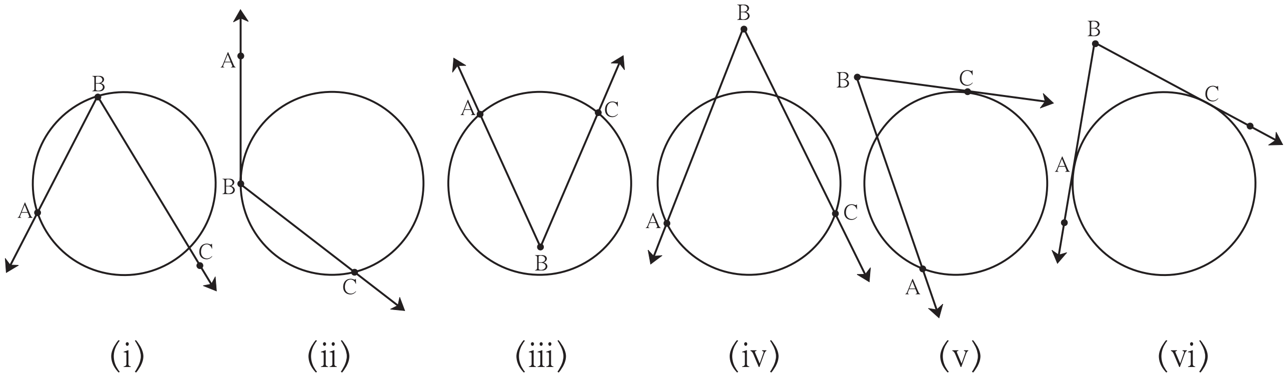


Fig. 3.43

In each figure, the arc of a circle that lies in the interior of the ∠ ABC is an arc intercepted by the ∠ ABC. The points of intersection of the circle and the angle are end points of that intercepted arc. Each side of the angle has to contain an end point of the arc.

In figures 3.43 (i), (ii) and (iii) only one arc is intercepted by that angle; and in (iv), (v) and (vi), two arcs are intercepted by the angle.

Also note that, only one side of the angle touches the circle in (ii) and (v), but in (vi) both sides of the angle touch the circle.

In figure 3.44, the arc is not intercepted arc, as arm BC does not contain any end point of the arc.

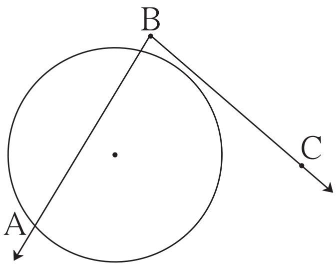


Fig. 3.44

Inscribed angle theorem

The measure of an inscribed angle is half of the measure of the arc intercepted by it.

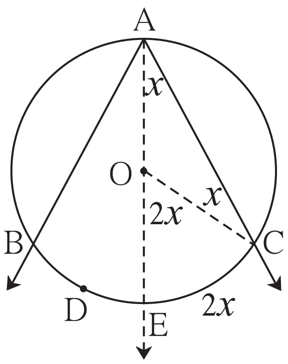


Fig. 3.45

Given : In a circle with centre O, ∠ BAC is inscribed in arc BAC. Arc BDC is intercepted by the angle.

To prove: $\angle BAC = \frac{1}{2}\, m(\text{arc BDC})$

Construction : Draw ray AO. It intersects the circle at E. Draw radius OC.

Proof : In Δ AOC,

side OA $\cong$ side OC ...... radii of the same circle.

$\therefore$ $\angle$ OAC = $\angle$ OCA ..... theorem of isosceles triangle.

Let $\angle$ OAC = $\angle$ OCA = x ...... (I)

Now, $\angle$ EOC = $\angle$ OAC + $\angle$ OCA .... exterior angle theorem of a triangle.

$= x^\circ + x^\circ = 2x^\circ$

But $\angle$ EOC is a central angle.

$\therefore$ m(arc EC) = $2x^\circ$ .... definition of measure of an arc ..... (II)

$\therefore$ from (I) and (II).

$\angle$ OAC = $\angle$ EAC = $\frac{1}{2}$ m(arc EC) ..... (III)

Similarly, drawing seg OB, we can prove $\angle$EAB = $\frac{1}{2}$ m(arc BE)...... (IV)

$\therefore$$\angle$EAC + $\angle$EAB = $\frac{1}{2}$ m(arc EC) + $\frac{1}{2}$ m(arc BE).... from (III) and (IV)

$\therefore$ $\angle$BAC = $\frac{1}{2}$ [m(arc EC) + m(arc BE)]

= $\frac{1}{2}$ [m(arc BEC)] = $\frac{1}{2}$ [m(arc BDC)] ..... (V)

Note that we have to consider three cases regarding the position of the centre of the circle and the inscribed angle. The centre of the circle lies (i) on one of the arms of the angle (ii) in the interior of the angle (iii) in the exterior of the angle. Out of these, first two are proved in (III) and (V). We will prove now the third one.

In figure 3.46,

$\angle$ BAC = $\angle$ BAE - $\angle$ CAE

= $\frac{1}{2}$ m(arc BCE) - $\frac{1}{2}$ m(arc CE)

...... from (III)

= $\frac{1}{2}$ [m(arc BCE) - m(arc CE)]

= $\frac{1}{2}$ [m(arc BC)] ...... (VI)

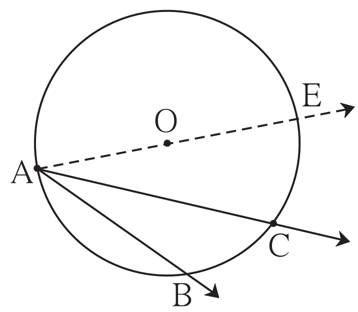


Fig. 3.46

The above theorem can also be stated as follows.

The measure of an angle subtended by an arc at a point on the circle is half of the measure of the angle subtended by the arc at the centre.

The corollaries of the above theorem can also be stated in similar language.

Corollaries of inscribed angle theorem

1. Angles inscribed in the same arc are congruent.

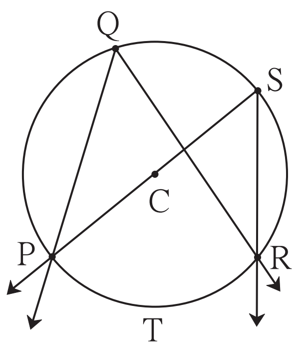


Fig. 3.47

Write 'given' and 'to prove' with the help of the figure 3.47.

Think of the answers of the following questions and write the proof.

(1) Which arc is intercepted by ∠ PQR ?

(2) Which arc is intercepted by ∠ PSR ?

(3) What is the relation between an inscribed angle and the arc intersepted by it ?

2. Angle inscribed in a semicircle is a right angle.

With the help of figure 3.48 write 'given', 'to prove' and 'the proof'.

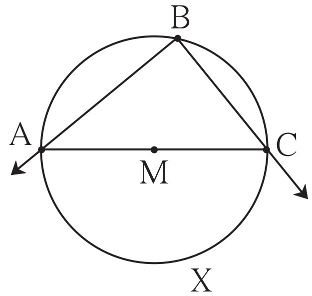


Fig. 3.48

Cyclic quadrilateral

If all vertices of a quadrilateral lie on the same circle then it is called a cyclic quadrilateral.

Theorem of cyclic quadrilateral

Theorem: **Opposite angles of a cyclic quadrilateral are supplementry.**

Fill in the blanks and complete the following proof.

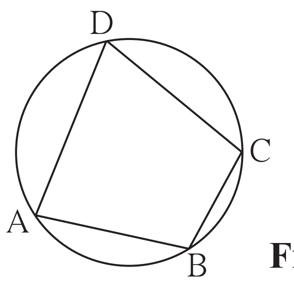


Fig. 3.49

Given : □ [ ] is cyclic.

To prove : ∠ B + ∠ D = [ ]

[ ] + ∠ C = 180°

Proof : Arc ABC is intercepted by the inscribed angle ∠ ADC.

∴ ∠ADC = $\frac{1}{2}$ [ ] .... (I)

Similarly, [ ] is an inscribed angle. It intercepts arc ADC.

$\therefore \; \square = \frac{1}{2} \, m(\text{arc ADC}) \; \ldots\ldots \; (\text{II})$

$\therefore \; m\angle ADC + \square = \frac{1}{2} \square + \frac{1}{2} \, m(\text{arc ADC}) \; \ldots\ldots$ from (I) & (II)

$= \frac{1}{2} \, [\square + m(\text{arc ADC})]$

$= \frac{1}{2} \times 360^\circ$ .... arc ABC and arc ADC constitute a complete circle.

$= \square$

Similarly we can prove, $\angle A + \angle C = \square$.

Corollary of cyclic quadrilateral theorem

An exterior angle of a cyclic quadrilateral is congruent to the angle opposite to its adjacent interior angle.

Write the proof of the theorem yourself.

Let's think.

In the above theorem, after proving $\angle B + \angle D = 180^\circ$, can you use another way to prove $\angle A + \angle C = 180^\circ$?

Converse of cyclic quadrilateral theorem

Theorem : If a pair of opposite angles of a quadrilateral is supplementary, the quadrilateral is cyclic.

Try to prove this theorem by 'indirect method'. From the above converse, we know that if opposite angles of a quadrilateral are supplementary then there is a circumcircle for the quadrilateral.

For every triangle there exists a circumcircle but there may not be a circumcircle for every quadrilateral.

The above converse gives us the condition to ensure the existence of circumcircle of a quadrilateral.

With one more condition four non-collinear points are concyclic. It is stated in the following theorem.

Theorem : If two points on a given line subtend equal angles at two distinct points which lie on the same side of the line, then the four points are concyclic.

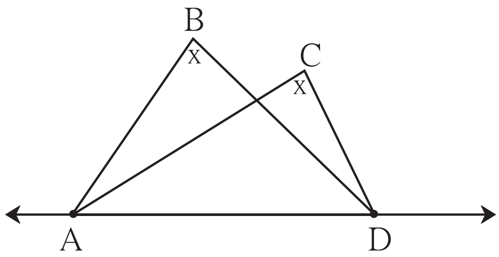


Fig. 3.50

Given : Points B and C lie on the same side of the line AD. $\angle$ ABD $\cong \angle$ ACD

To prove: Points A, B, C, D are concyclic. (That is, $\square$ABCD is cyclic.)

This theorem can be proved by 'indirect method'.

Let's think.

The above theorem is converse of a certain theorem. State it.

Solved Examples

Ex. (1) In figure 3.51, chord LM $\cong$ chord LN

$\angle$ L = 35° find

(i) m(arc MN)

(ii) m(arc LN)

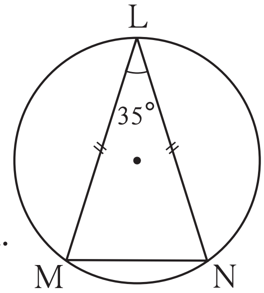


Fig. 3.51

Solution : (i) $\angle L = \frac{1}{2}\, m(\text{arc MN})$ ......inscribed angle theorem.

$\therefore\ 35 = \frac{1}{2}\, m(\text{arc MN})$

$\therefore\ 2 \times 35 = m(\text{arc MN}) = 70°$

(ii) m(arc MLN) = 360° − m(arc MN) ...... definition of measure of arc

= 360° − 70° = 290°

Now, chord LM $\cong$ chord LN

$\therefore$ arc LM $\cong$ arc LN

but m(arc LM) + m(arc LN) = m(arc MLN) = 290° ..... arc addition property

$m(\text{arc LM}) = m(\text{arc LN}) = \frac{290°}{2} = 145°$

or, (ii) chord LM $\cong$ chord LN

$\therefore\ \angle M = \angle N$ ...... isosceles trinangle theorem.

$\therefore\ 2\angle M = 180° - 35° = 145°$

$\therefore\ \angle M = \frac{145°}{2}$

Now, $m(\text{arc LN}) = 2 \times \angle M$

$$= 2 \times \frac{145^\circ}{2} = 145^\circ$$

Ex. (2) In figure 3.52, chords PQ and RS intersect at T.

(i) Find $m(\text{arc SQ})$ if $m\angle STQ = 58^\circ$, $m\angle PSR = 24^\circ$.

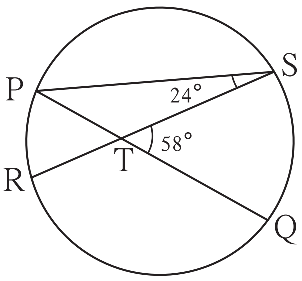


Fig. 3.52

(ii) Verify,

$\angle STQ = \frac{1}{2}[m(\text{arc PR}) + m(\text{arc SQ})]$

(iii) Prove that :

$\angle STQ = \frac{1}{2}[m(\text{arc PR}) + m(\text{arc SQ})]$

for any measure of $\angle STQ$.

(iv) Write in words the property in (iii).

Solution : (i) $\angle SPQ = \angle SPT = 58^\circ - 24^\circ = 34^\circ$ .... exterior angle theorem.

$m(\text{arc QS}) = 2\,\angle SPQ = 2 \times 34^\circ = 68^\circ$

(ii) $m(\text{arc PR}) = 2\,\angle PSR = 2 \times 24^\circ = 48^\circ$

Now, $\frac{1}{2}[m(\text{arc PR}) + m(\text{arc SQ})] = \frac{1}{2}[48 + 68]$

$$= \frac{1}{2} \times 116 = 58^\circ$$

$$= \angle STQ$$

(iii) Fill in the blanks and complete the proof of the above property.

$\angle STQ = \angle SPQ + \square$ .... exterior angle theorem of a triangle

$= \frac{1}{2} m(\text{arc SQ}) + \square$ .... inscribed angle theorem

$= \frac{1}{2}[\square + \square]$

(iv) If two chords of a circle intersect each other in the interior of a circle then the measure of the angle between them is half the sum of measures of arcs intercepted by the angle and its opposite angle.

Ex. (3) Prove that, if two lines containing chords of a circle intersect each other outside the circle, then the measure of angle between them is half the difference in measures of the arcs intercepted by the angle.

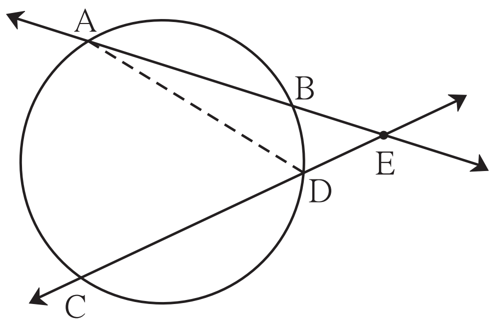


Fig. 3.53

Given : Chords AB and CD intersect at E in the exterior of the circle.

To prove: $\angle AEC = \frac{1}{2}[m(\text{arc AC}) - m(\text{arc BD})]$

Construction: Draw seg AD.

Consider angles of Δ AED and its exterior angle and write the proof.

Remember this !

(1) The measure of an inscribed angle is half the measure of the arc intercepted by it.

(2) Angles inscribed in the same arc are congruent.

(3) Angle inscribed in a semicircle is a right angle.

(4) If all vertices of a quadrilateral lie on the same circle then the quadrilateral is called a cyclic quadrilateral.

(5) Opposite angles of a cyclic quadrilateral are supplementary.

(6) An exterior angle of a cyclic quadrilateral is congruent to the angle opposite to its adjacent interior angle.

(7) If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

(8) If two points on a given line subtend equal angles at two different points which lie on the same side of the line, then those four points are concyclic.

(9) In figure 3.54,

(i) $\angle AEC = \frac{1}{2}\ [m(\text{arc AC}) + m(\text{arc DB})]$

(ii) $\angle CEB = \frac{1}{2}\ [m(\text{arc AD}) + m(\text{arc CB})]$

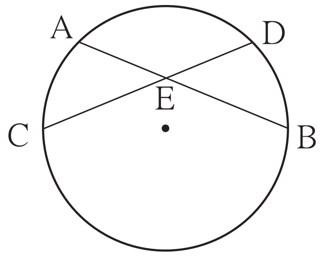


Fig. 3.54

(10) In figure 3.55,

$\angle BED = \frac{1}{2}[m(\text{arc }BD) - m(\text{arc }AC)]$

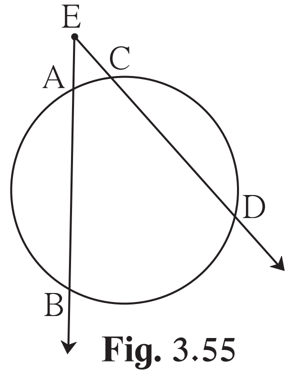


Fig. 3.55

Practice set 3.4

1. In figure 3.56, in a circle with centre O, length of chord AB is equal to the radius of the circle. Find measure of each of the following.

(1) $\angle AOB$ (2) $\angle ACB$

(3) arc AB (4) arc ACB.

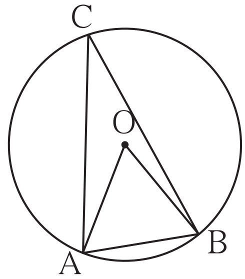


Fig. 3.56

2. In figure 3.57, □ PQRS is cyclic. side PQ ≅ side RQ. $\angle PSR = 110°$, Find–

(1) measure of $\angle PQR$

(2) m(arc PQR)

(3) m(arc QR)

(4) measure of $\angle PRQ$

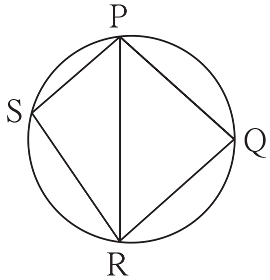


Fig. 3.57

3. □ MRPN is cyclic, $\angle R = (5x - 13)°$, $\angle N = (4x + 4)°$. Find measures of $\angle R$ and $\angle N$.

4. In figure 3.58, seg RS is a diameter of the circle with centre O. Point T lies in the exterior of the circle. Prove that $\angle RTS$ is an acute angle.

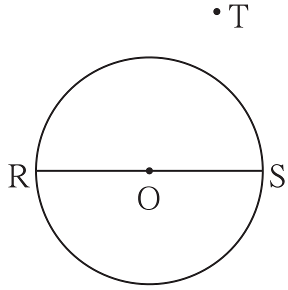


Fig. 3.58

5. Prove that, any rectangle is a cyclic quadrilateral.

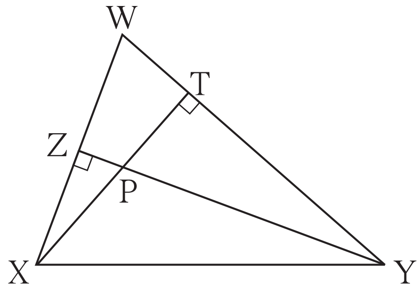


Fig. 3.59

6. In figure 3.59, altitudes YZ and XT of Δ WXY intersect at P. Prove that,

(1) □ WZPT is cyclic.

(2) Points X, Z, T, Y are concyclic.

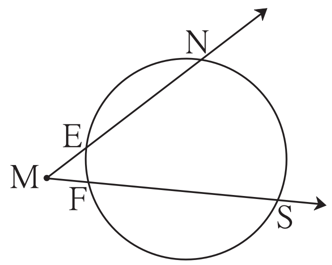


Fig. 3.60

7. In figure 3.60, m(arc NS) = 125°, m(arc EF) = 37°, find the measure $\angle$NMS.

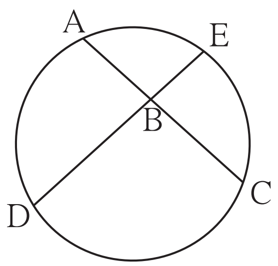


Fig. 3.61

8. In figure 3.61, chords AC and DE intersect at B. If $\angle$ ABE = 108°, m(arc AE) = 95°, find m(arc DC) .

Let's learn.

Activity :

Draw a circle as shown in figure 3.62. Draw a chord AC. Take any point B on the circle. Draw inscribed $\angle$ABC, measure it and note the measure.

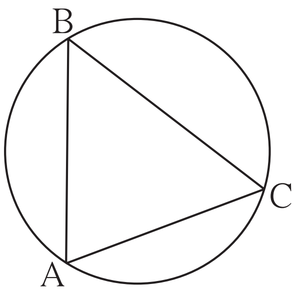


Fig. 3.62

Now as shown in figure 3.63, draw a tangent CD of the same circle, measure angle $\angle$ACD and note the measure.

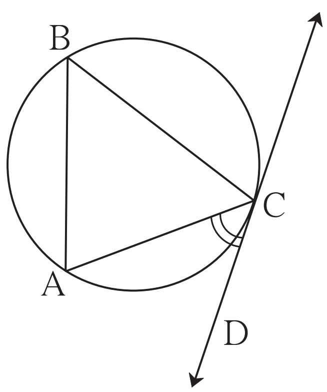


Fig. 3.63

You will find that $\angle ACD = \angle ABC$.

You know that $\angle ABC = \frac{1}{2}\, m(\text{arc } AC)$

From this we get $\angle ACD = \frac{1}{2}\, m(\text{arc } AC)$.

Now we will prove this property.

Theorem of angle between tangent and secant

If an angle has its vertex on the circle, its one side touches the circle and the other intersects the circle in one more point, then the measure of the angle is half the measure of its intercepted arc.

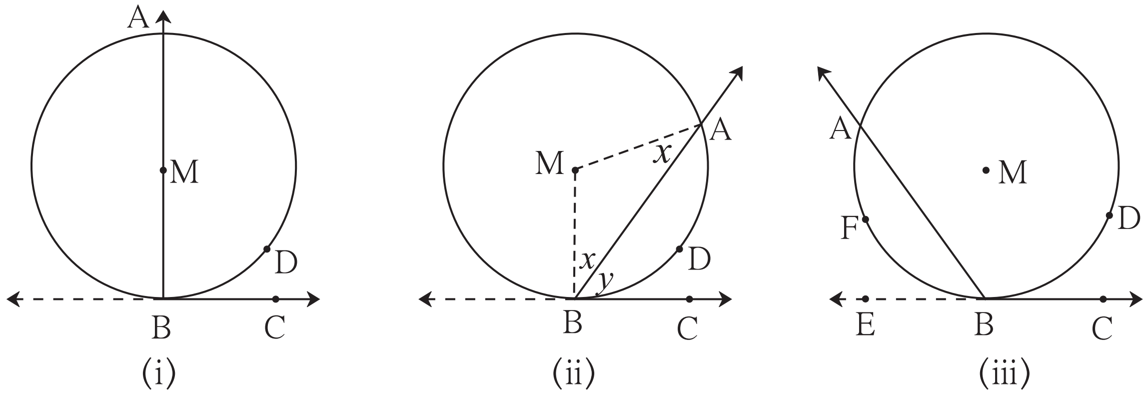


Fig. 3.64

Given : Let $\angle ABC$ be an angle, where vertex B lies on a circle with centre M. Its side BC touches the circle at B and side BA intersects the circle at A. Arc ADB is intercepted by $\angle ABC$.

To prove: $\angle ABC = \frac{1}{2}\, m(\text{arc } ADB)$

Proof : Consider three cases.

(1) In figure 3.64 (i), the centre M lies on the arm BA of $\angle ABC$,

$\angle ABC = \angle MBC = 90^\circ$ ..... tangent theorem (I)

arc ADB is a semicircle.

$\therefore\ m(\text{arc } ADB) = 180^\circ$ ..... definition of measure of arc (II)

From (I) and (II)

$\angle ABC = \frac{1}{2}\, m(\text{arc } ADB)$

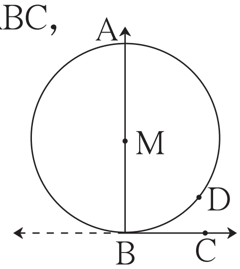


Fig. 3.64(i)

(2) In figure 3.64 (ii) centre M lies in the exterior of $\angle ABC$,

Draw radii MA and MB.

Now, $\angle MBA = \angle MAB$ ..... isosceles triangle theorem

$\angle MBC = 90^\circ$ ..... tangent theorem..... (I) let $\angle$ MBA = $\angle$ MAB = x and $\angle$ ABC = y.

$\angle$ AMB = 180 − $(x + x)$ = 180 − $2x$

$\angle$ MBC = $\angle$ MBA + $\angle$ ABC = $x + y$

$\therefore$ $x + y = 90°$ $\quad\therefore$ $2x + 2y = 180°$

In Δ AMB, $2x$ + $\angle$ AMB = 180°

$\therefore$ $2x + 2y = 2x$ + $\angle$ AMB

$\therefore$ $2y$ = $\angle$ AMB

$\therefore$ y = $\angle$ ABC = $\frac{1}{2}$ $\angle$ AMB = $\frac{1}{2}$ m(arc ADB)

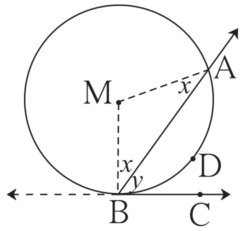


Fig. 3.64(ii)

(3) With the help of fig 3.64 (iii),

Fill in the blanks and write proof.

Ray ______ is the opposite ray of ray BC.

Now, $\angle$ABE = $\frac{1}{2}$ m(   ) ...... proved in (ii).

$\therefore$ 180 − ______ = $\frac{1}{2}$ m(arc AFB) ..... linear pair

= $\frac{1}{2}$ [360 − m(______)]

$\therefore$ 180 − $\angle$ABC = 180 − $\frac{1}{2}$ m(arc ADB)

$\therefore$ − $\angle$ABC = − $\frac{1}{2}$ m(______)

$\therefore$ $\angle$ABC = $\frac{1}{2}$ m(arc ADB)

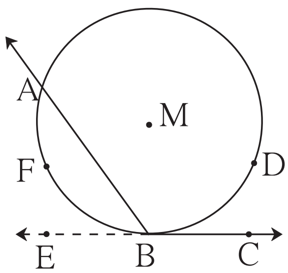


Fig. 3.64(iii)

Alternative statement of the above theorem.

In the figure 3.65, line AB is a secant and line BC is a tangent. The arc ADB is intercepted by $\angle$ ABC. Chord AB divides the circle in two parts. These are opposite arcs of each other.

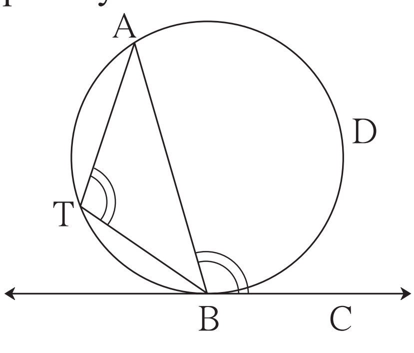


Fig. 3.65

Now take any point T on the arc opposite to arc ADB .

From the above theorem,

$\angle$ ABC = $\frac{1}{2}$ m (arc ADB) = $\angle$ ATB.

$\therefore$ the angle between a tangent of a circle and a chord drawn from the point of contact is congruent to the angle inscribed in the arc opposite to the arc intercepted by that angle.

Converse of theorem of the angle between tangent and secant

A line is drawn from one end point of a chord of a circle and if the angle between the chord and the line is half the measure of the arc intercepted by that angle then that line is a tangent to the circle.

In figure 3.66,

If $\angle PQR = \frac{1}{2}\ m(\text{arc } PSQ)$,

[or $\angle PQT = \frac{1}{2}\ m(\text{arc } PUQ)$]

then line TR is a tangent to the circle.

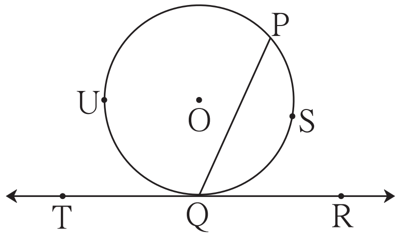


Fig. 3.66

This property is used in constructing a tangent to the given circle.

An indirect proof of this converse can be given.

Theorem of internal division of chords

Suppose two chords of a circle intersect each other in the interior of the circle, then the product of the lengths of the two segments of one chord is equal to the product of the lengths of the two segments of the other chord.

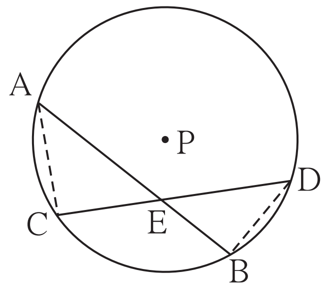


Fig. 3.67

Given : Chords AB and CD of a circle with centre P intersect at point E.

To prove: AE × EB = CE × ED

Construction : Draw seg AC and seg DB.

Proof : In Δ CAE and Δ BDE,

$\angle AEC \cong \angle DEB$ ..... opposite angles

$\angle CAE \cong \angle BDE$ ..... angles inscribed in the same arc

$\therefore \Delta CAE \sim \Delta BDE$ ..... AA test

$\therefore \frac{AE}{DE} = \frac{CE}{BE}$ ..... corresponding sides of similar triangles

$\therefore AE \times EB = CE \times ED$

Let's think.

We proved the theorem by drawing seg AC and seg DB in figure 3.67, Can the theorem be proved by drawing seg AD and segCB, instead of seg AC and seg DB?

For more information

In figure 3.67 point E divides the chord AB into segments AE and EB. AE × EB is the area of a rectangle having sides AE and EB. Similarly E divides CD into segments CE and ED. CE × ED is the area of a rectangle of sides CE and ED. We have proved that AE × EB = CE × ED.

So the above theorem can be stated as, 'If two chords of a circle intersect in the interior of a circle then the area of the rectangle formed by the segments of one chord is equal to the area of similar rectangle formed by the other chord.'

Theorem of external division of chords

If secants containing chords AB and CD of a circle intersect outside the circle in point E, then AE × EB = CE × ED.

Write 'given' and 'to prove' with the help of the statement of the theorem and figure 3.68.

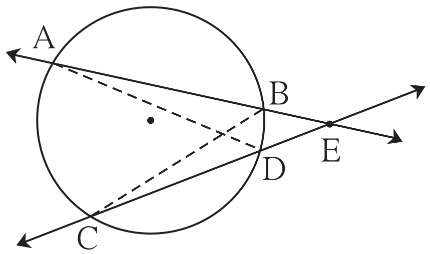


Fig. 3.68

Construction : Draw seg AD and seg BC.

Fill in the blanks and complete the proof.

Proof : In Δ ADE and Δ CBE,

∠ AED ≅ $\boxed{}$ .... common angle

∠ DAE ≅ ∠ BCE ....($\boxed{}$)

∴ Δ ADE ~ $\boxed{}$ ....($\boxed{}$)

$\therefore \frac{(AE)}{\boxed{}} = \frac{\boxed{}}{\boxed{}}$ .... corresponding sides of similar triangles

∴ $\boxed{}$ = CE × ED

Tangent secant segments theorem

Point E is in the exterior of a circle. A secant through E intersects the circle at points A and B, and a tangent through E touches the circle at point T, then $EA \times EB = ET^2$.

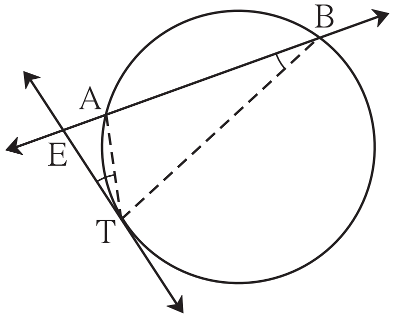


Fig. 3.69

Write 'given' and 'to prove' with reference to the statement of the theorem.

Construction : Draw seg TA and seg TB.

Proof : In Δ EAT and Δ ETB,

$\angle$ AET $\cong$ $\angle$ TEB .... common angle

$\angle$ ETA $\cong$ $\angle$ EBT... tangent secant theorem

$\therefore$ Δ EAT ~ Δ ETB ..... AA similarity

$\therefore \frac{ET}{EB} = \frac{EA}{ET}$ ..... corresponding sides

$\therefore EA \times EB = ET^2$

Remember this !

(1) In figure 3.70,

AE × EB = CE × ED

This property is known as theorem of chords intersecting inside the circle.

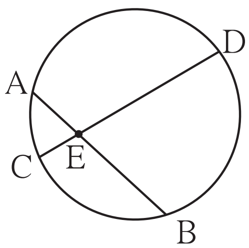


Fig. 3.70

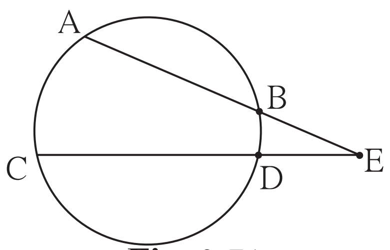


Fig. 3.71

(2) In figure 3.71,

AE × EB = CE × ED

This property is known as theorem of chords intersecting outside the circle.

(3) In figure 3.72,

$EA \times EB = ET^2$

This property is known as tangent secant segments theorem.

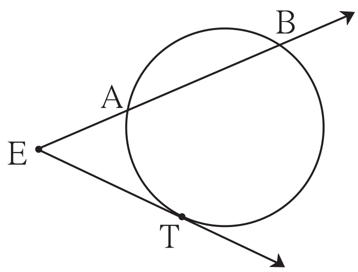


Fig. 3.72

Solved Examples

Ex. (1) In figure 3.73, seg PS is a tangent segment. Line PR is a secant. If PQ = 3.6, QR = 6.4, find PS.

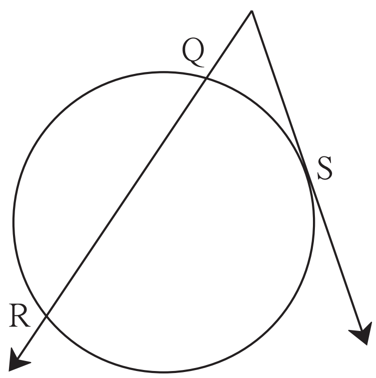


Fig. 3.73

Solution : $PS^2 = PQ \times PR$ .... tangent secant segments theorem

$= PQ \times (PQ + QR)$

$= 3.6 \times [3.6 + 6.4]$

$= 3.6 \times 10$

$= 36$

$\therefore PS = 6$

Ex. (2) In figure 3.74, chord MN and chord RS intersect each other at point P. If PR = 6, PS = 4, MN = 11 find PN.

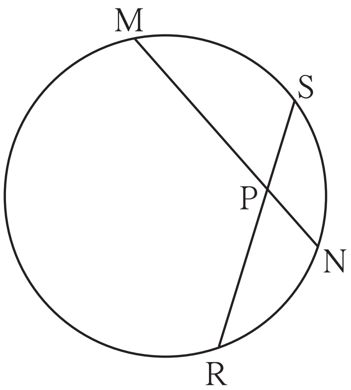


Fig. 3.74

Solution : By theorem on interesecting chords,

$PN \times PM = PR \times PS$... (I)

let PN = x. $\therefore PM = 11 - x$

substituting the values in (I),

$x\,(11 - x) = 6 \times 4$

$\therefore\ 11x - x^2 - 24 = 0$

$\therefore\ x^2 - 11x + 24 = 0$

$\therefore\ (x - 3)\,(x - 8) = 0$

$\therefore\ x - 3 = 0$ or $x - 8 = 0$

$\therefore\ x = 3$ or $x = 8$

$\therefore$ PN = 3 or PN = 8

Ex. (3) In figure 3.75, two circles intersect each other in points X and Y. Tangents drawn from a point M on line XY touch the circles at P and Q.

Prove that, seg PM $\cong$ seg QM.

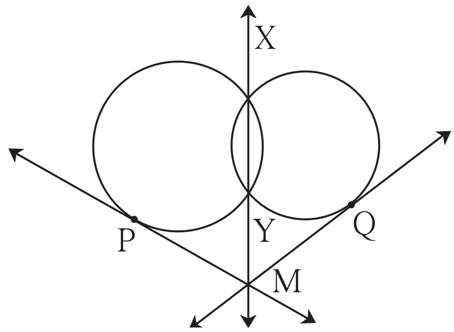


Fig. 3.75

Solution : Fill in the blanks and write proof.

Line MX is a common .... of the two circles.

$\therefore PM^2 = MY \times MX$ ..... (I)

Similarly .... = ... $\times$ .... , tangent secant segment theorem ......(II)

$\therefore$ From (I) and (II) ...... = QM^2

$\therefore$ PM = QM

seg PM $\cong$ seg QM

Ex. (4)

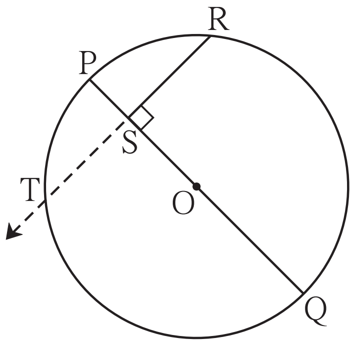


Fig. 3.76

In figure 3.76, seg PQ is a diameter of a circle with centre O. R is any point on the circle.

seg RS $\perp$ seg PQ.

Prove that, SR is the geometric mean of PS and SQ.

[That is, $SR^2 = PS \times SQ$]

Solution : Write the proof with the help of the following steps.

(1) Draw ray RS. It intersects the circle at T.

(2) Show that RS = TS.

(3) Write a result using theorem of intersection of chords inside the circle.

(4) Using RS = TS complete the proof.

Let's think.

(1) In figure 3.76, if seg PR and seg RQ are drawn, what is the nature of Δ PRQ?

(2) Have you previously proved the property proved in example (4) ?

Practice set 3.5

1. In figure 3.77, ray PQ touches the circle at point Q. PQ = 12, PR = 8, find PS and RS.

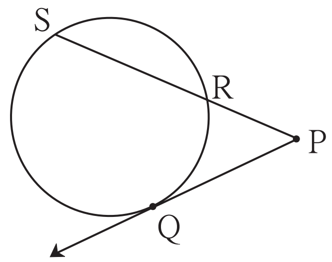


Fig. 3.77

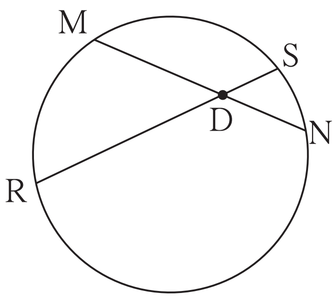


Fig. 3.78

2. In figure 3.78, chord MN and chord RS intersect at point D.

(1) If RD = 15, DS = 4, MD = 8 find DN

(2) If RS = 18, MD = 9, DN = 8 find DS

3. In figure 3.79, O is the centre of the circle and B is a point of contact. seg OE ⊥ seg AD, AB = 12, AC = 8, find

(1) AD (2) DC (3) DE.

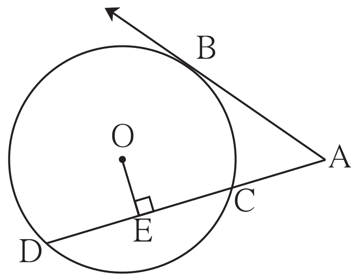


Fig. 3.79

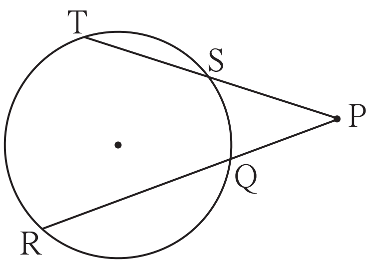


Fig. 3.80

4. In figure 3.80, if PQ = 6, QR = 10, PS = 8 find TS.

5. In figure 3.81, seg EF is a diameter and seg DF is a tangent segment. The radius of the circle is r. Prove that, DE × GE = $4r^2$

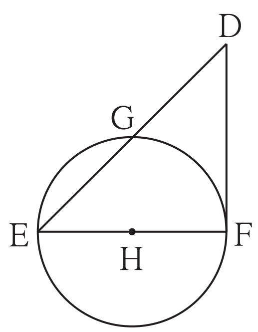


Fig. 3.81

Problem set 3

1. Four alternative answers for each of the following questions are given. Choose the correct alternative.

(1) Two circles of radii 5.5 cm and 3.3 cm respectively touch each other. What is the distance between their centers ?

(A) 4.4 cm (B) 8.8 cm (C) 2.2 cm (D) 8.8 or 2.2 cm

(2) Two circles intersect each other such that each circle passes through the centre of the other. If the distance between their centres is 12, what is the radius of each circle ?

(A) 6 cm (B) 12 cm (C) 24 cm (D) can't say

(3) A circle touches all sides of a parallelogram. So the parallelogram must be a, ...

(A) rectangle (B) rhombus (C) square (D) trapezium

(4) Length of a tangent segment drawn from a point which is at a distance 12.5 cm from the centre of a circle is 12 cm, find the diameter of the circle.

(A) 25 cm (B) 24 cm (C) 7 cm (D) 14 cm

(5) If two circles are touching externally, how many common tangents of them can be drawn?

(A) One (B) Two (C) Three (D) Four

(6) $\angle$ACB is inscribed in arc ACB of a circle with centre O. If $\angle$ACB = 65°, find m(arc ACB).

(A) 65° (B) 130° (C) 295° (D) 230°

(7) Chords AB and CD of a circle intersect inside the circle at point E. If AE = 5.6, EB = 10, CE = 8, find ED.

(A) 7 (B) 8 (C) 11.2 (D) 9

(8) In a cyclic □ABCD, twice the measure of $\angle$ A is thrice the measure of $\angle$C. Find the measure of $\angle$C ?

(A) 36 (B) 72 (C) 90 (D) 108

9)★ Points A, B, C are on a circle, such that m(arc AB) = m(arc BC) = 120°. No point, except point B, is common to the arcs. Which is the type of Δ ABC ?

(A) Equilateral triangle (B) Scalene triangle

(C) Right angled triangle (D) Isosceles triangle

(10) Seg XZ is a diameter of a circle. Point Y lies in its interior. How many of the following statements are true ?

(i) It is not possible that $\angle$ XYZ is an acute angle.

(ii) $\angle$ XYZ can't be a right angle.

(iii) $\angle$ XYZ is an obtuse angle.

(iv) Can't make a definite statement for measure of $\angle$ XYZ.

(A) Only one (B) Only two (C) Only three (D) All

2. Line l touches a circle with centre O at point P. If radius of the circle is 9 cm, answer the following.

(1) What is d(O, P) = ? Why ?

(2) If d(O, Q) = 8 cm, where does the point Q lie ?

(3) If d(PQ) = 15 cm, How many locations of point R are line on line l ? At what distance will each of them be from point P ?

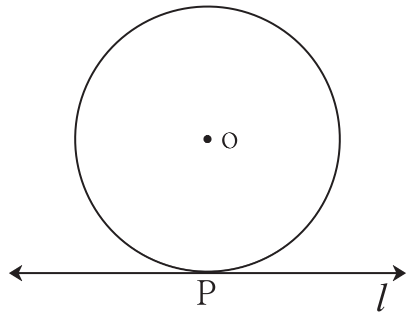


Fig. 3.82

3. In figure 3.83, M is the centre of the circle and seg KL is a tangent segment.

If MK = 12, KL = $6\sqrt{3}$ then find -

(1) Radius of the circle.

(2) Measures of $\angle$ K and $\angle$ M.

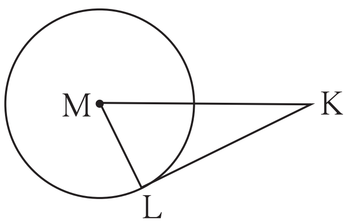


Fig. 3.83

4. In figure 3.84, O is the centre of the circle. Seg AB, seg AC are tangent segments. Radius of the circle is r and l(AB) = r, Prove that, $\square$ABOC is a square.

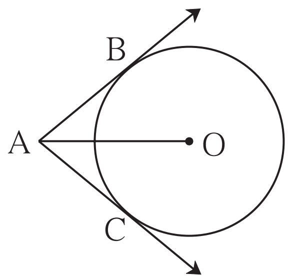


Fig. 3.84

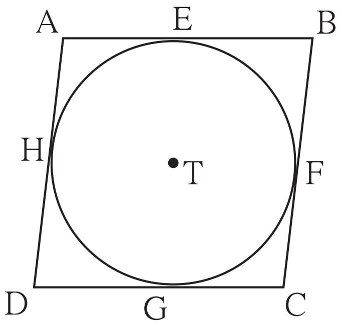


Fig. 3.85

5. In figure 3.85, □ABCD is a parallelogram. It circumscribes the circle with cnetre T. Point E, F, G, H are touching points. If AE = 4.5, EB = 5.5, find AD.

6. In figure 3.86, circle with centre M touches the circle with centre N at point T. Radius RM touches the smaller circle at S. Radii of circles are 9 cm and 2.5 cm. Find the answers to the following questions hence find the ratio MS:SR.

(1) Find the length of segment MT

(2) Find the length of seg MN

(3) Find the measure of ∠ NSM.

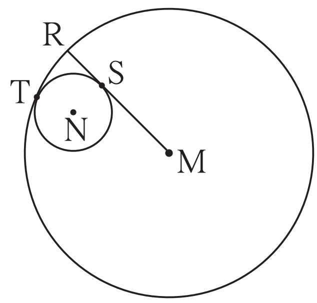


Fig. 3.86

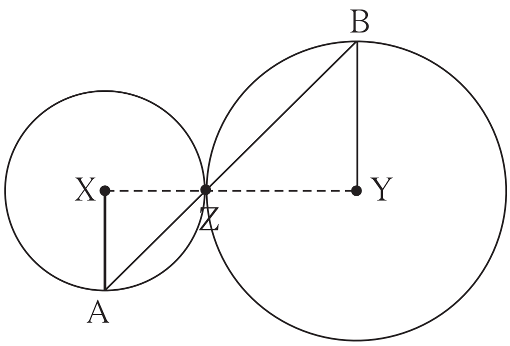


Fig. 3.87

7. In the adjoining figure circles with centres X and Y touch each other at point Z. A secant passing through Z intersects the circles at points A and B respectively. Prove that, radius XA || radius YB.

Fill in the blanks and complete the proof.

Construction : Draw segments XZ and ....

Proof :By theorem of touching circles, points X, Z, Y are ....

∴ ∠ XZA ≅ .... opposite angles

Let ∠ XZA = ∠ BZY = a ..... (I)

Now, seg XA ≅ seg XZ .... (..........)

∴ ∠ XAZ = ... = a .... (isosceles triangle theorem) (II)

similarly, seg YB ≅ .... (..........)

∴ ∠ BZY = ... = a .... (..........) (III)

$\therefore$ from (I), (II), (III),

$\angle$ XAZ = ...

$\therefore$ radius XA || radius YB .... (..........)

8. In figure 3.88, circles with centres X and Y touch internally at point Z . Seg BZ is a chord of bigger circle and it itersects smaller circle at point A.
Prove that, seg AX || seg BY.

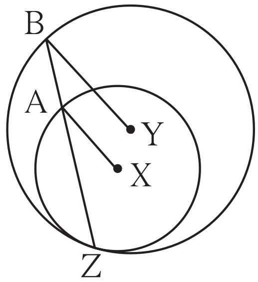


Fig. 3.88

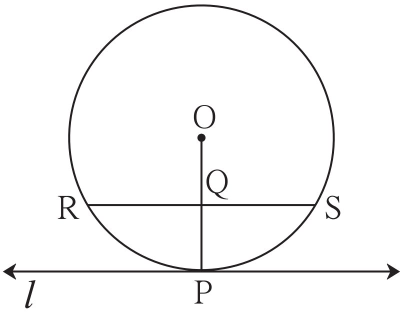


Fig. 3.89

9. In figure 3.89, line *l* touches the circle with centre O at point P. Q is the mid point of radius OP. RS is a chord through Q such that chords RS || line *l*. If RS = 12 find the radius of the circle.

10★. In figure 3.90, seg AB is a diameter of a circle with centre C. Line PQ is a tangent, which touches the circle at point T.
seg AP ⊥ line PQ and seg BQ ⊥ line PQ.
Prove that, seg CP ≅ seg CQ.

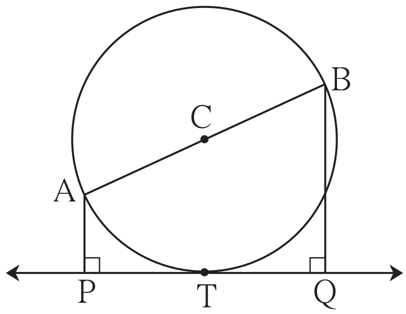


Fig. 3.90

11★. Draw circles with centres A, B and C each of radius 3 cm, such that each circle touches the other two circles.

12★. Prove that any three points on a circle cannot be collinear.

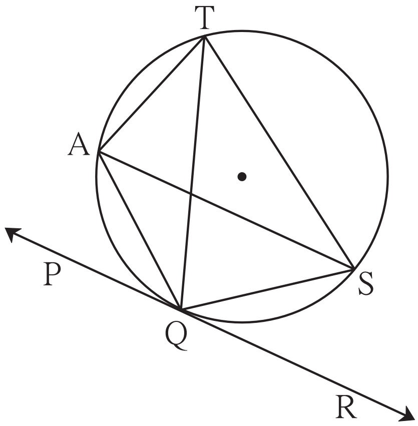


Fig. 3.91

13. In figure 3.91, line PR touches the circle at point Q. Answer the following questions with the help of the figure.

(1) What is the sum of $\angle$TAQ and $\angle$ TSQ ?

(2) Find the angles which are congruent to $\angle$AQP.

(3) Which angles are congruent to $\angle$QTS ?

(4) $\angle$TAS = 65°, find the measure of $\angle$TQS and arc TS.

(5) If $\angle$AQP = 42°and $\angle$SQR = 58° find measure of $\angle$ATS.

14. In figure 3.92, O is the centre of a circle, chord PQ $\cong$ chord RS

If $\angle$ POR = 70°

and (arc RS) = 80°, find –

(1) m(arc PR)

(2) m(arc QS)

(3) m(arc QSR)

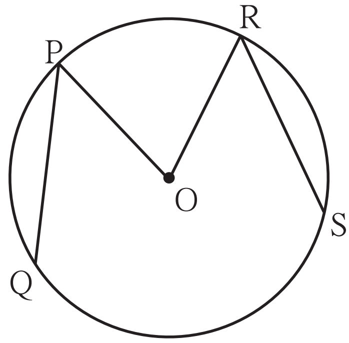


Fig. 3.92

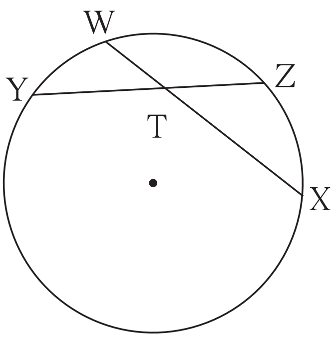


Fig. 3.93

15. In figure 3.93, m(arc WY) = 44°, m(arc ZX) = 68°, then

(1) Find the measure of $\angle$ ZTX.

(2) If WT = 4.8, TX = 8.0, YT = 6.4, find TZ.

(3) If WX = 25, YT = 8, YZ = 26, find WT.

16. In figure 3.94,

(1) m(arc CE) = 54°,

m(arc BD) = 23°, find measure of ∠CAE.

(2) If AB = 4.2, BC = 5.4,

AE = 12.0, find AD

(3) If AB = 3.6, AC = 9.0,

AD = 5.4, find AE

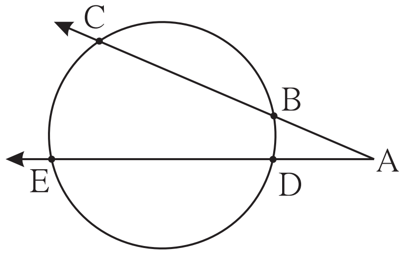


Fig. 3.94

17. In figure 3.95, chord EF || chord GH. Prove that, chord EG ≅ chord FH.

Fill in the blanks and write the proof.

Proof : Draw seg GF.

∠ EFG = ∠ FGH .... [ ] (I)

∠ EFG = [ ] .... inscribed angle theorem (II)

∠ FGH = [ ] .... inscribed angle theorem (III)

∴ m(arc EG) = [ ] from (I), (II), (III).

chord EG ≅ chord FH .... [ ]

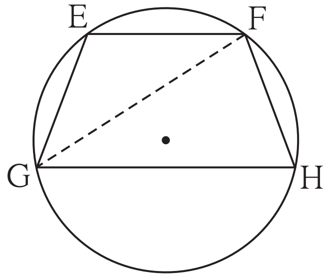


Fig. 3.95

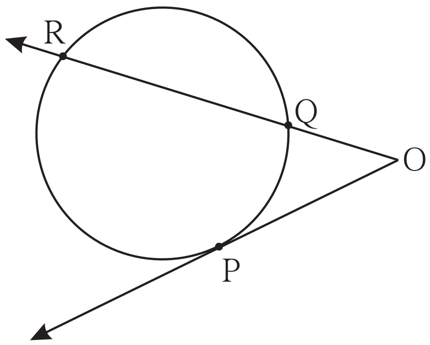


Fig. 3.96

18. In figure 3.96 P is the point of contact.

(1) If m(arc PR) = 140°,

∠ POR = 36°,

find m(arc PQ)

(2) If OP = 7.2, OQ = 3.2,

find OR and QR

(3) If OP = 7.2, OR = 16.2,

find QR.

19. In figure 3.97, circles with centres C and D touch internally at point E. D lies on the inner circle. Chord EB of the outer circle intersects inner circle at point A. Prove that, seg EA ≅ seg AB.

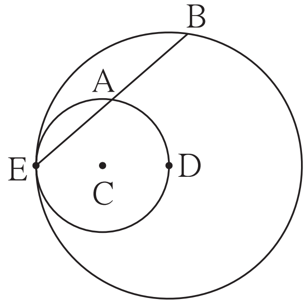


Fig. 3.97

20. In figure 3.98, seg AB is a diameter of a circle with centre O . The bisector of $\angle$ ACB intersects the circle at point D. Prove that, seg AD $\cong$ seg BD.
Complete the following proof by filling in the blanks.

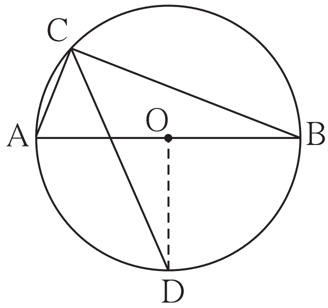


Fig. 3.98

Proof : Draw seg OD.

$\angle$ ACB = ______ .... angle inscribed in semicircle

$\angle$ DCB = ______ .... CD is the bisector of $\angle$ C

m(arc DB) = ______ .... inscribed angle theorem

$\angle$ DOB = ______ .... definition of measure of an arc (I)

seg OA $\cong$ seg OB .... ______ (II)

$\therefore$ line OD is ______ of seg AB .... From (I) and (II)

$\therefore$ seg AD $\cong$ seg BD

21. In figure 3.99, seg MN is a chord of a circle with centre O. MN = 25, L is a point on chord MN such that ML = 9 and d(O,L) = 5. Find the radius of the circle.

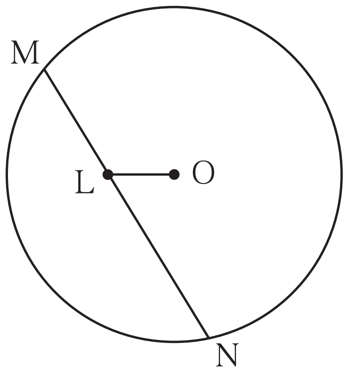


Fig. 3.99

22★. In figure 3.100, two circles intersect each other at points S and R. Their common tangent PQ touches the circle at points P, Q.
Prove that, $\angle$ PRQ + $\angle$ PSQ = 180°

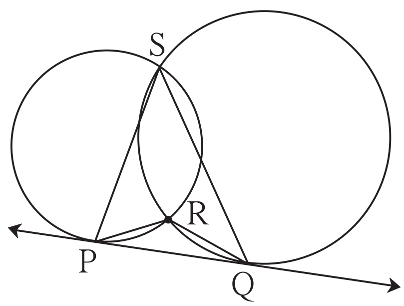


Fig. 3.100

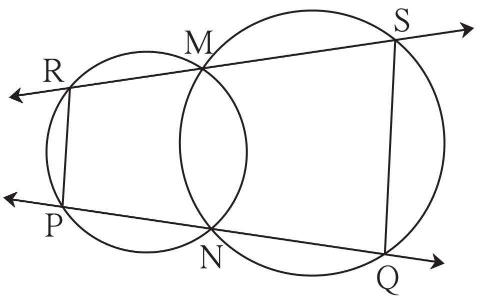


Fig. 3.101

23★**.** In figure 3.101, two circles intersect at points M and N. Secants drawn through M and N intersect the circles at points R, S and P, Q respectively.

Prove that : seg SQ || seg RP.

24★**.** In figure 3.102, two circles intersect each other at points A and E. Their common secant through E intersects the circles at points B and D. The tangents of the circles at points B and D intersect each other at point C.

Prove that □ ABCD is cyclic.

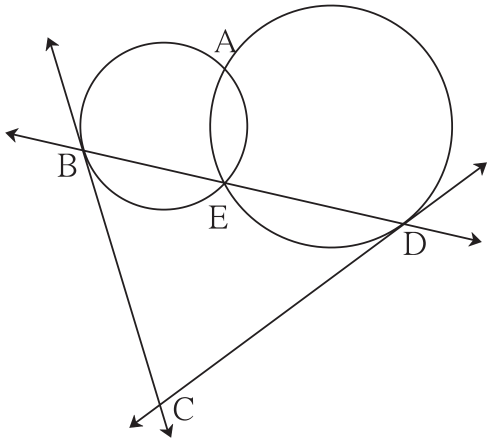


Fig. 3.102

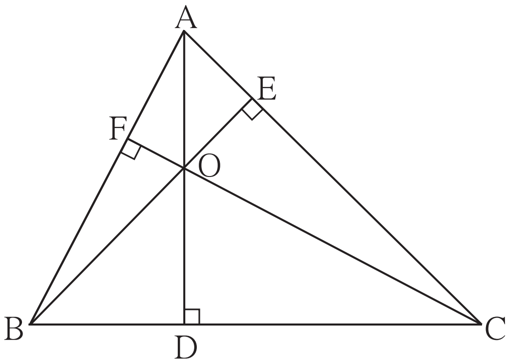


Fig. 3.103

25★**.** In figure 3.103, seg AD ⊥ side BC, seg BE ⊥ side AC, seg CF ⊥ side AB. Ponit O is the orthocentre. Prove that , point O is the incentre of Δ DEF.

ICT Tools or Links

Use the geogebra to verify the properties of chords and tangents of a circle.

4 Geometric Constructions

Let's study.

- Construction of a triangle similar to the given triangle
  - ✻ To construct a triangle, similar to the given triangle, bearing the given ratio with the sides of the given triangle.
    - (i) When vertices are distinct
    - (ii) When one vertex is common
- Construction of a tangent to a circle.
  - ✻ To construct a tangent at a point on the circle.
    - (i) Using centre of the circle.
    - (ii) Without using the centre of the circle.
  - ✻ To construct tangents to the given circle from a point outside the circle.

Let's recall.

In the previous standard you have learnt the following constructions. Let us recall those constructions.

- To construct a line parallel to a given line and passing through a given point outside the line.
- To construct the perpendicular bisector of a given line segment.
- To construct a triangle whose sides are given.
- To divide a given line segment into given number of equal parts
- To divide a line segment in the given ratio.
- To construct an angle congruent to the given angle.

In the ninth standard you have carried out the activity of preparing a map of surroundings of your school. Before constructing a building we make its plan. The surroundings of a school and its map, the building and its plan are similar to each other. We need to draw similar figures in Geography, architecture, machine drawing etc. A triangle is the simplest closed figure. We shall learn how to construct a triangle similar to the given triangle.

Let's learn.

Construction of Similar Triangle

To construct a triangle similar to the given triangle, satisfying the condition of given ratio of corresponding sides.

The corresponding sides of similar triangles are in the same proportion and the corresponding angles of these triangles are equal. Using this property, a triangle which is similar to the given triangle can be constructed.

Ex. (1) Δ ABC ~ Δ PQR, in Δ ABC, AB = 5.4 cm, BC = 4.2 cm, AC = 6.0 cm. AB: PQ = 3:2. Construct Δ ABC and Δ PQR.

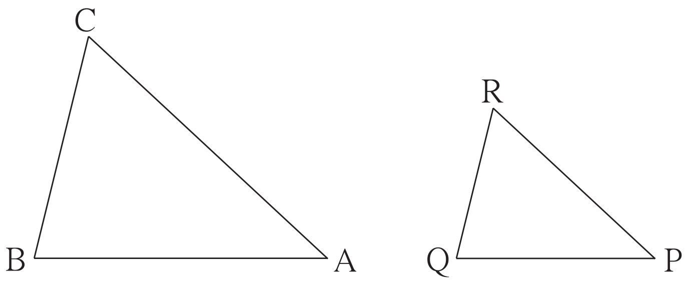


Fig. 4.1
Rough Figure

Construct Δ ABC of given measure.

Δ ABC and Δ PQR are similar.

$\therefore$ their corresponding sides are proportional.

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{3}{2} \quad \text{..........} \quad \text{(I)}$$

As the sides AB, BC, AC are known, we can find the lengths of sides PQ, QR, PR.

Using equation [I]

$$\frac{5.4}{PQ} = \frac{4.2}{QR} = \frac{6.0}{PR} = \frac{3}{2}$$

$\therefore$ PQ = 3.6 cm, QR = 2.8 cm and PR = 4.0 cm

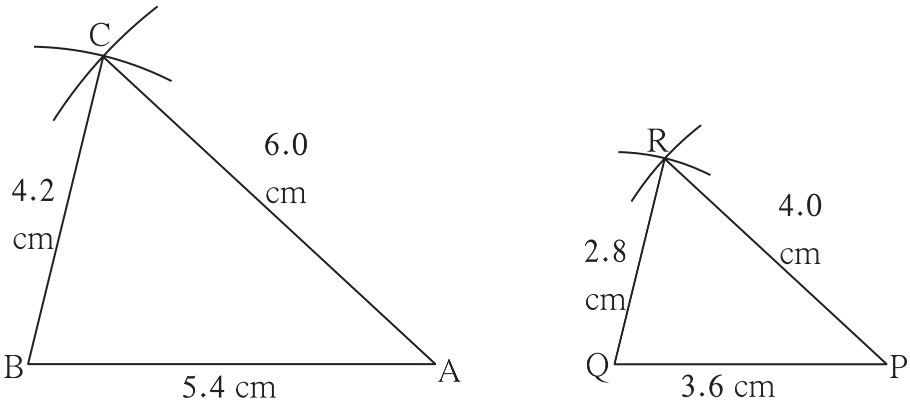


Fig. 4.2

For More Information

While drawing the triangle similar to the given triangle, sometimes the lengths of the sides we obtain by calculation are not easily measureable by a scale. In such a situation we can use the construction 'To divide the given segment in the given number of eqaul parts'.

For example, if length of side AB is $\frac{11.6}{3}$ cm, then by dividing the line segment of length 11.6 cm in three equal parts, we can draw segment AB.

If we know the lengths of all sides of Δ PQR, we can construct Δ PQR.

In the above example (1) there was no common vertex in the given triangle and the triangle to be constructed. If there is a common vertex, it is convenient to follow the method in the following example.

Ex.(2) Construct any Δ ABC. Construct Δ A′BC′ such that AB : A′B = 5:3 and $\Delta ABC \sim \Delta A'BC'$

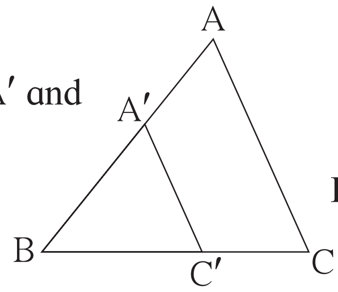


Fig. 4.3

Rough Figure

Analysis : As shown in fig 4.3 , let the points B, A, A′ and B, C, C′ be collinear.

$\Delta ABC \sim \Delta A'BC'$ $\therefore$ $\angle ABC = \angle A'BC'$

$$\frac{AB}{A'B} = \frac{BC}{BC'} = \frac{AC}{A'C'} = \frac{5}{3}$$

$\therefore$ sides of Δ ABC are longer than corresponding sides of Δ A′BC′ .

$\therefore$ the length of side BC′ will be equal to 3 parts out of 5 equal parts of side BC. So if we construct Δ ABC, point C′ will be on the side BC, at a distance equal to 3 parts from B. Now A′ is the point of intersection of AB and a line through C′, parallel to CA.

$$\frac{BA'}{BA} = \frac{BC'}{BC} = \frac{3}{5} \text{ i.e, } \frac{BA}{BA'} = \frac{BC}{BC'} = \frac{5}{3} \text{ ...... Taking inverse}$$

Steps of construction :

(1) Construct any Δ ABC.

(2) Divide segment BC in 5 equal parts.

(3) Name the end point of third part of seg BC as C′ ∴ BC′ = $\frac{3}{5}$ BC

(4) Now draw a line parallel to AC through C′. Name the point where the parallel line intersects AB as A′.

(5) Δ A′BC′ is the required trinangle similar to Δ ABC

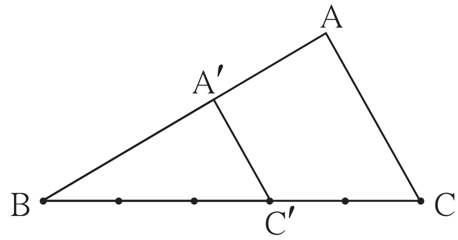


Fig. 4.4

Note : To divide segment BC, in five equal parts, it is convenient to draw a ray from B, on the side of line BC in which point A does not lie.

Take points T_1, T_2, T_3, T_4, T_5 on the ray such that

$BT_1 = T_1T_2 = T_2T_3 = T_3T_4 = T_4T_5$

Join T_5C and draw lines parallel to T_5C through T_1, T_2, T_3, T_4 .

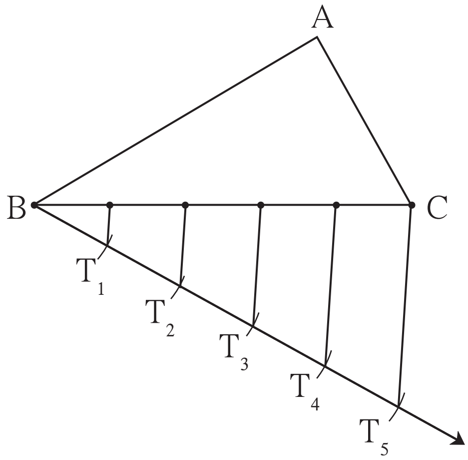


Fig. 4.5

Let's think

Δ A′BC′ can also be constructed as shown in the adjoining figure.

What changes do we have to make in steps of construction in that case ?

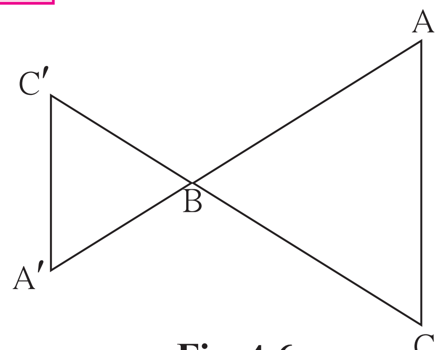


Fig. 4.6

Ex. (3) Construct Δ A′BC′ similar to Δ ABC such that AB:A′B = 5:7

Analysis : Let points B, A, A′ as well as points B, C, C′ be collinear.

Δ ABC ~ Δ A′BC′ and AB : A′B = 5:7

$\therefore$ sides of Δ ABC are smaller than sides of Δ A′BC′

and $\angle ABC \cong \angle A'BC'$

Let us draw a rough figure with these considerations. Now $\frac{BC}{BC'} = \frac{5}{7}$

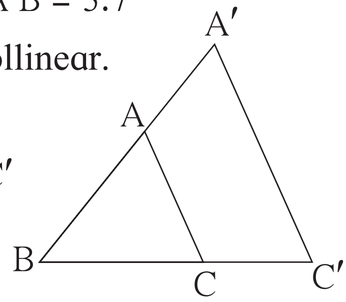


Fig. 4.7
Rough Figure

$\therefore$ If seg BC is divided into 5 equal parts, then seg BC′ will be 7 times each part of seg BC.

$\therefore$ let us divide side BC of Δ ABC in 5 equal parts and locate point C′ at a distance equal to 7 such parts from B on ray BC. A line through point C′ parallel to seg AC is drawn it will intersect ray BA at point A′. According to the basic proportionality theorem we will get Δ A′BC′ as described.

Steps of construction :

(1) Construct any Δ ABC.

(2) Divide segment BC into 5 five equal parts. Fix point C′ on ray BC such that length of BC′ is seven times of each equal part of seg BC

(3) Draw a line parallel to side AC, through C′. Name the point of intersection of the line and ray BA as A′.

We get the required Δ A′BC′ similar to Δ ABC.

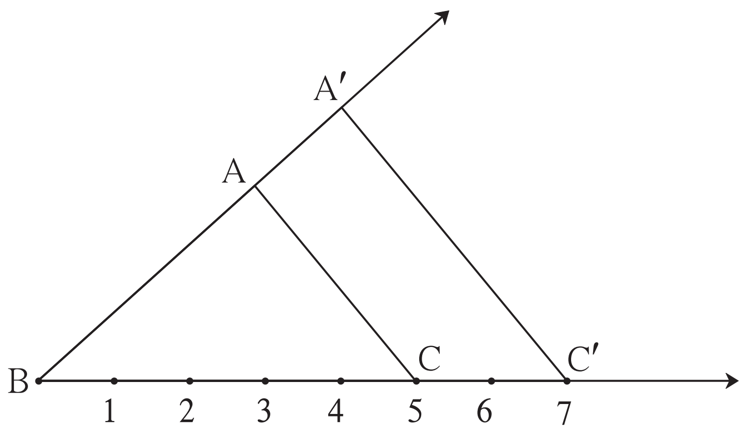


Fig. 4.8

Practice set 4.1

1. Δ ABC ~ Δ LMN. In Δ ABC, AB = 5.5 cm, BC = 6 cm, CA = 4.5 cm. Construct Δ ABC and Δ LMN such that $\frac{BC}{MN} = \frac{5}{4}$.

2. Δ PQR ~ Δ LTR. In Δ PQR, PQ = 4.2 cm, QR = 5.4 cm, PR = 4.8 cm. Construct Δ PQR and Δ LTR, such that $\frac{PQ}{LT} = \frac{3}{4}$.

3. Δ RST ~ ΔXYZ. In Δ RST, RS = 4.5 cm, $\angle$RST = 40°, ST = 5.7 cm Construct Δ RST and Δ XYZ, such that $\frac{RS}{XY} = \frac{3}{5}$.

4. Δ AMT ~ Δ AHE. In Δ AMT, AM = 6.3 cm, $\angle$TAM = 50°, AT = 5.6 cm. $\frac{AM}{AH} = \frac{7}{5}$. Construct Δ AHE.

Construction of a tangent to a circle at a point on the circle

(i) Using the centre of the circle.

Analysis :

Suppose we want to construct a tangent l passing through a point P on the circle with centre C. We shall use the property that a line perpendicular to the radius at its outer end is a tangent to the circle. If CP is a radius with point P on the circle, line l through P and perpendicular to CP is the tangent at P. For this we will use the construction of drawing a perpendicular to a line through a point on it.

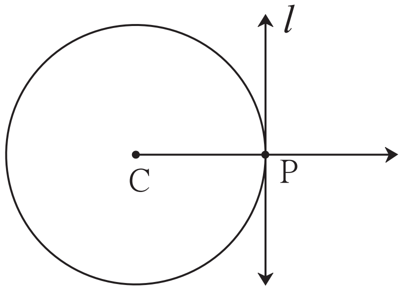


Fig. 4.9

For convenience we shall draw ray CP

Steps of construction

(1) Draw a circle with centre C.
Take any point P on the circle.

(2) Draw ray CP.

(3) Draw line l perpendicular to ray CX through point P.
Line l is the required tangent to the circle at point 'P'.

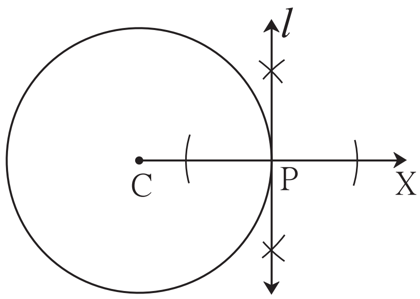


Fig. 4.10

ii) Without using the centre of the circle.

Example: Construct a circle of any radius. Take any point C on it. Construct a tangent to the circle without using centre of the circle.

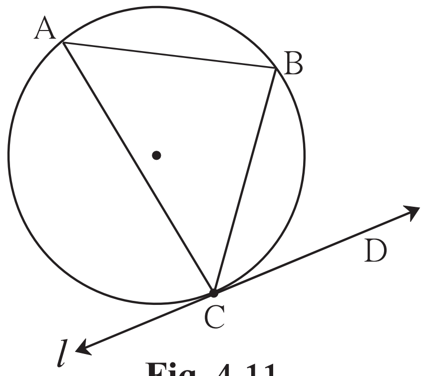


Fig. 4.11

Analysis :

As shown in the figure, let line l be the tangent to the circle at point C. Line CB is a chord and $\angle$ CAB is an inscribed angle. Now by tangent- secant angle theorem, $\angle CAB \cong \angle BCD$.

By converse of tangent- secant theorem, if we draw the line CD such that, $\angle CAB \cong \angle BCD$, then it will be the required tangent.

Steps of Construction :

(1) Draw a circle of a suitable radius. Take any point C on it.

(2) Draw chord CB and an inscribed $\angle CAB$.

(3) With the centre A and any convenient radius draw an arc intersecting the sides of $\angle BAC$ in points M and N.

(4) Using the same radius and centre C, draw an arc intersecting the chord CB at point R.

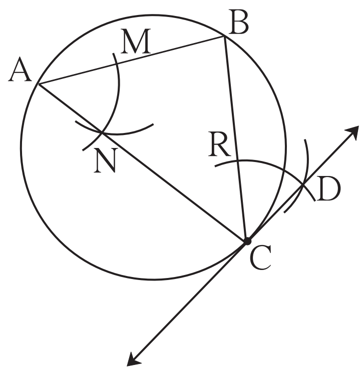


Fig. 4.12

(5) Taking the radius equal to d(MN) and centre R, draw an arc intersecting the arc drawn in the previous step. Let D be the point of intersection of these arcs. Draw line CD. Line CD is the required tangent to the circle.

Note that $\angle$ MAN and $\angle$ BCD in the above figure are congruent. If we draw seg MN and seg RD, then Δ MAN and Δ RCD are congruent by SSS test.

$\therefore \angle MAN \cong \angle BCD$

To construct tangents to a circle from a point outside the circle.

Analysis :

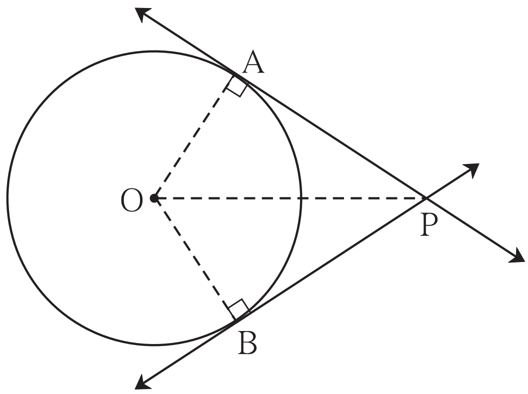


Fig. 4.13

As shown in the figure let P be a point in the exterior of the circle. Let PA and PB be the tangents to the circle with the centre O, touching the circle in points A and B respectively. So if we find points A and B on the circle, we can construct the tangents PA and PB. If OA and OB are the radii of the circle, then OA ⊥ line PA and seg OB ⊥ line PB.

Δ OAP and OBP are right angled triangles and seg OP is their common hypotenuse. If we draw a circle with diameter OP, then the points where it intersects the circle with centre O, will be the positions of points A and B respectively, because angle inscribed in a semicircle is a right angle.

Steps of Construction

(1) Construct a circle of any radius with centre O.

(2) Take any point P in the exterior of the circle.

(3) Draw segment OP. Draw perpendicular bisector of seg OP to get its midpoint M.

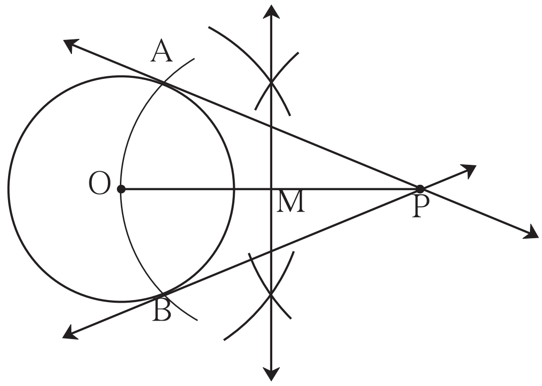


Fig. 4.14

(4) Draw a circle with radius OM and centre M

(5) Name the points of intersection of the two circles as A and B.

(6) Draw line PA and line PB.

Practice set 4.2

1. Construct a tangent to a circle with centre P and radius 3.2 cm at any point M on it.

2. Draw a circle of radius 2.7 cm. Draw a tangent to the circle at any point on it.

3. Draw a circle of radius 3.6 cm. Draw a tangent to the circle at any point on it without using the centre.

4. Draw a circle of radius 3.3 cm Draw a chord PQ of length 6.6 cm. Draw tangents to the circle at points P and Q. Write your obeservation about the tangents.

5. Draw a circle with radius 3.4 cm. Draw a chord MN of length 5.7 cm in it. construct tangents at point M and N to the circle.
6. Draw a circle with centre P and radius 3.4 cm. Take point Q at a distance 5.5 cm from the centre. Construct tangents to the circle from point Q.
7. Draw a circle with radius 4.1 cm. Construct tangents to the circle from a point at a distance 7.3 cm from the centre.

Problem set 4

1. Select the correct alternative for each of the following questions.

(1) The number of tangents that can be drawn to a circle at a point on the circle is ....

(A) 3 (B) 2 (C) 1 (D) 0

(2) The maximun number of tangents that can be drawn to a circle from a point out side it is ....

(A) 2 (B) 1 (C) one and only one (D) 0

(3) If $\Delta ABC \sim \Delta PQR$ and $\frac{AB}{PQ} = \frac{7}{5}$, then ....

(A) Δ ABC is bigger. (B) Δ PQR is bigger.

(C) Both triangles will be equal. (D) Can not be decided.

2. Draw a circle with centre O and radius 3.5 cm. Take point P at a distance 5.7 cm from the centre. Draw tangents to the circle from point P.
3. Draw any circle. Take any point A on it and construct tangent at A without using the centre of the circle.
4. Draw a circle of diameter 6.4 cm. Take a point R at a distance equal to its diameter from the centre. Draw tangents from point R.
5. Draw a circle with centre P. Draw an arc AB of $100°$ measure. Draw tangents to the circle at point A and point B.
6. Draw a circle of radius 3.4 cm and centre E. Take a point F on the circle. Take another point A such that E-F-A and FA = 4.1 cm. Draw tangents to the circle from point A.
7. Δ ABC $\sim$ ΔLBN. In Δ ABC, AB = 5.1cm, $\angle B = 40°$, BC=4.8 cm, $\frac{AC}{LN} = \frac{4}{7}$. Construct Δ ABC and ΔLBN.
8. Construct Δ PYQ such that, PY = 6.3 cm, YQ = 7.2 cm, PQ = 5.8 cm. If $\frac{YZ}{YQ} = \frac{6}{5}$, then construct Δ XYZ similar to Δ PYQ.

PUN6L7

5 Co-ordinate Geometry

Let's study.

- Distance formula
- Section formula
- Slope of a line

Let's recall.

We know how to find the distance between any two points on a number line.

If co-ordinates of points P,Q and R are $-1, -5$ and 4 respectively then find the length of seg PQ, seg QR.

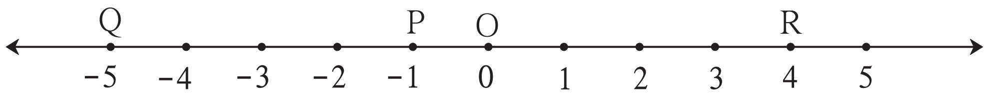


Fig. 5.1

If x_1 and x_2 are the co-ordinates of points A and B and $x_2 > x_1$ then length of seg $AB = d(A,B) = x_2 - x_1$

As shown in the figure, co-ordinates of points P,Q and R are $-1, -5$ and 4 respectively.

$\therefore\ d(P, Q) = (-1)-(-5) = -1 + 5 = 4$

and $d(Q, R) = 4 - (-5) = 4 + 5 = 9$

Using the same concept we can find the distance between two points on the same axis in XY-plane.

Let's learn.

(1) To find distance between any two points on an axis .

Two points on an axis are like two points on the number line. Note that points on the X-axis have co-ordinates such as $(2, 0)$, $(\frac{-5}{2}, 0)$, $(8, 0)$. Similarly points on the Y-axis have co-ordinates such as $(0, 1)$, $(0, \frac{17}{2})$, $(0, -3)$. Part of the X-axis which shows negative co-ordinates is OX′ and part of the Y-axis which shows negative co-ordinates is OY′.

i) To find distance between two points on X-axis.

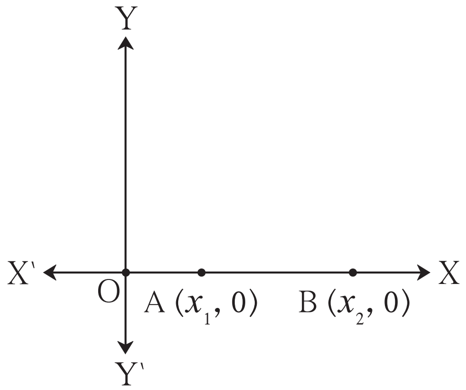


Fig. 5.2

In the above figure, points $A(x_1, 0)$ and $B(x_2, 0)$ are on X- axis such that, $x_2 > x_1$

$\therefore$ d(A, B) = $x_2 - x_1$

ii) To find distance between two points on Y-axis.

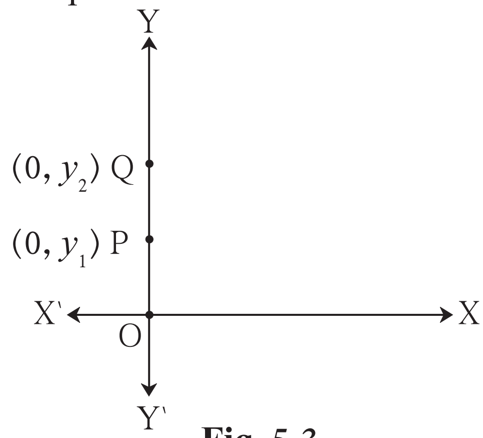


Fig. 5.3

In the above figure, points $P(0, y_1)$ and $Q(0, y_2)$ are on Y- axis such that, $y_2 > y_1$

$\therefore$ d(P,Q) = $y_2 - y_1$

(2) To find the distance between two points if the segment joining these points is parallel to any axis in the XY plane.

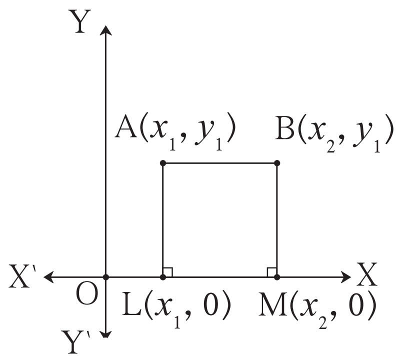


Fig. 5.4

i) In the figure, seg AB is parallel to X- axis.

$\therefore$ y co-ordinates of points A and B are equal

Draw seg AL and seg BM perpendicular to X-axis

$\therefore$ □ ABML is a rectangle.

$\therefore$ AB = LM

But, LM = $x_2 - x_1$

$\therefore$ d(A,B) = $x_2 - x_1$

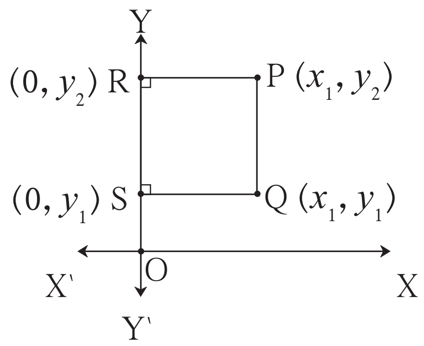


Fig. 5.5

ii) In the figure seg PQ is parallel to Y- axis.

$\therefore$ x co-ordinates of points P and Q are equal

Draw seg PR and seg QS perpendicular to Y-axis.

$\therefore$ □ PQSR is a rectangle

$\therefore$ PQ = RS

But, RS = $y_2 - y_1$

$\therefore$ d(P,Q) = $y_2 - y_1$

Activity:

In the figure, seg AB || Y-axis and seg CB || X-axis. Co-ordinates of points A and C are given.

To find AC, fill in the boxes given below.

Δ ABC is a right angled triangle.

According to Pythagoras theorem,

$(AB)^2 + (BC)^2 =$ [ ]

We will find co-ordinates of point B to find the lengths AB and BC,

CB || X- axis ∴ y co-ordinate of B = [ ]

BA || Y- axis ∴ x co-ordinate of B = [ ]

AB = [3] − [ ] = [ ]   BC = [ ] − [ ] = [4]

∴ AC^2 = [ ] + [ ] = [ ]   ∴ AC = [$\sqrt{17}$]

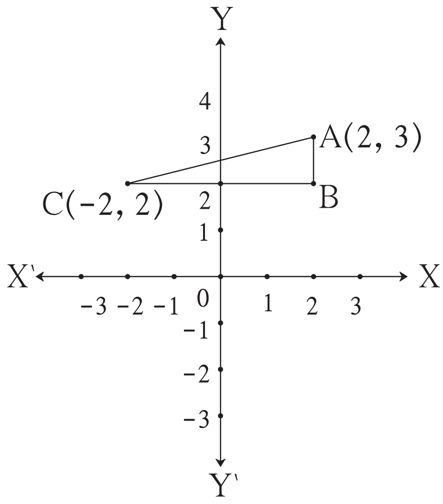


Fig. 5.6

Let's learn.

Distance formula

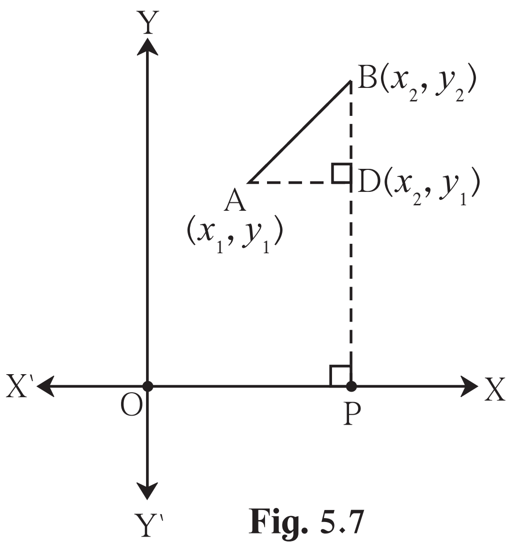


Fig. 5.7

In the figure 5.7, A(x_1, y_1) and B(x_2, y_2) are any two points in the XY plane.

From point B draw perpendicular BP on X-axis.

Similarly from point A draw perpendicular AD on seg BP.

seg BP is parallel to Y-axis.

∴ the x co-ordinate of point D is x_2.

seg AD is parallel to X-axis.

∴ the y co-ordinate of point D is y_1.

∴ AD = d(A, D) = $x_2 - x_1$; BD= d(B, D) = $y_2 - y_1$

In right angled triangle Δ ABD, $AB^2 = AD^2 + BD^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is known as distance formula.

Note that, $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

In the previous activity, we found the lengths of seg AB and seg AC and then used Pythagoras theorem to find the length of seg AC.
Now we will use distance formula to find AC.

A(2, 3) and C(−2, 2) is given

Let A(x_1, y_1) and C(x_2, y_2).

$x_1 = 2, y_1 = 3, x_2 = -2, y_2 = 2$

$$AC = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(-2-2)^2+(2-3)^2}$$

$$= \sqrt{(-4)^2+(-1)^2}$$

$$= \sqrt{16+1}$$

$$= \sqrt{17}$$

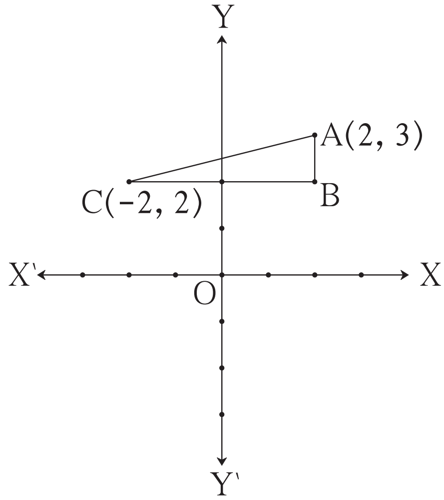


Fig. 5.8

seg AB || Y−axis and seg BC || X−axis.

∴ co-ordinates of point B are (2, 2).

$$\therefore AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(2-2)^2+(2-3)^2} = \sqrt{0+1} = 1$$

$$BC = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(-2-2)^2+(2-2)^2} = \sqrt{(-4)^2+0} = 4$$

In the Figure 5.1, distance between points P and Q is found as (−1) − (−5) = 4. In XY- plane co-ordinates of these points are (−1, 0) and (−5, 0). Verify that, using the distance formula we get the same answer.

Remember this!

- Co-ordinates of origin are (0, 0). Hence if co-ordinates of point P are (x, y) then d(O, P) = $\sqrt{x^2+y^2}$.
- If points P(x_1, y_1), Q(x_2, y_2) lie on the XY plane then

  d(P, Q) = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

  that is, $PQ^2 = (x_2-x_1)^2+(y_2-y_1)^2 = (x_1-x_2)^2+(y_1-y_2)^2$

Solved Examples

Ex. (1) Find the distance between the points P(−1, 1) and Q (5,−7) .

Solution : Suppose co-ordinates of point P are (x_1, y_1) and of point Q are (x_2, y_2).

$x_1 = -1,\quad y_1 = 1,\quad x_2 = 5,\quad y_2 = -7$

According to distance formula, $d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{[5-(-1)]^2 + [(-7)-1]^2}$$

$$= \sqrt{(6)^2 + (-8)^2}$$

$$= \sqrt{36+64}$$

$$d(P, Q) = \sqrt{100} = 10$$

$\therefore$ distance between points P and Q is 10.

Ex. (2) Show that points A(−3, 2), B(1, −2) and C(9, −10) are collinear.

Solution : If the sum of any two distances out of d(A, B), d(B, C) and d(A, C) is equal to the third , then the three points A, B and C are collinear.

$\therefore$ we will find d(A, B), d(B, C) and d(A, C).

Co-ordinates of A	Co-ordinates of B	Distance formula
(−3, 2)	(1, −2)	$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
(x_1, y_1)	(x_2, y_2)	

$\therefore\ d(A, B) = \sqrt{[1-(-3)]^2 + [(-2)-2]^2}$ .... from distance formula

$$= \sqrt{(1+3)^2 + (-4)^2}$$

$$= \sqrt{16+16}$$

$= \sqrt{32} = 4\sqrt{2}$ ....(I)

$d(B, C) = \sqrt{(9-1)^2 + (-10+2)^2}$

$= \sqrt{64+64} = 8\sqrt{2}$ ....(II)

and $d(A, C) = \sqrt{(9+3)^2 + (-10-2)^2}$

$= \sqrt{144+144} = 12\sqrt{2}$ ....(III)

$\therefore$ from(I), (II) and (III) $4\sqrt{2} + 8\sqrt{2} = 12\sqrt{2}$

$\therefore$ d(A, B) + d(B, C) = d(A, C)

$\therefore$ Points A, B, C are collinear.

Ex. (3) Verify, whether points P(6, −6), Q(3, −7) and R(3, 3) are collinear.

Solution : PQ $= \sqrt{(6-3)^2+(-6+7)^2}$ ....by distance formula

$= \sqrt{(3)^2+(1)^2} = \sqrt{10}$ .... (I)

QR $= \sqrt{(3-3)^2+(-7-3)^2}$

$= \sqrt{(0)^2+(-10)^2} = \sqrt{100}$ .... (II)

PR $= \sqrt{(3-6)^2+(3+6)^2}$

$= \sqrt{(-3)^2+(9)^2} = \sqrt{90}$ .... (III)

From I, II and III out of $\sqrt{10}$, $\sqrt{100}$ and $\sqrt{90}$, $\sqrt{100}$ is the largest number.

Now we will verify whether $\left(\sqrt{100}\right)$ and $\left(\sqrt{10}+\sqrt{90}\right)$ are equal.

For this compare $\left(\sqrt{100}\right)^2$ and $\left(\sqrt{10}+\sqrt{90}\right)^2$.

You will see that $\left(\sqrt{10}+\sqrt{90}\right) > \left(\sqrt{100}\right)$ $\therefore$ PQ + PR $\neq$ QR

$\therefore$ points P(6, −6), Q(3, −7) and R(3, 3) are not collinear.

Ex. (4) Show that points (1, 7), (4, 2), (−1, −1) and (−4, 4) are vertices of a square.

Solution : In a quadrilateral, if all sides are of equal length and both diagonals are of equal length, then it is a square.

$\therefore$ we will find lengths of sides and diagonals by using the distance formula.

Suppose, A(1, 7), B(4, 2), C(−1, −1) and D(−4,4) are the given points.

AB $= \sqrt{(1-4)^2+(7-2)^2} = \sqrt{9+25} = \sqrt{34}$

BC $= \sqrt{(4+1)^2+(2+1)^2} = \sqrt{25+9} = \sqrt{34}$

CD $= \sqrt{(-1+4)^2+(-1-4)^2} = \sqrt{9+25} = \sqrt{34}$

DA $= \sqrt{(1+4)^2+(7-4)^2} = \sqrt{25+9} = \sqrt{34}$

AC $= \sqrt{(1+1)^2+(7+1)^2} = \sqrt{4+64} = \sqrt{68}$

BD $= \sqrt{(4+4)^2+(2-4)^2} = \sqrt{64+4} = \sqrt{68}$

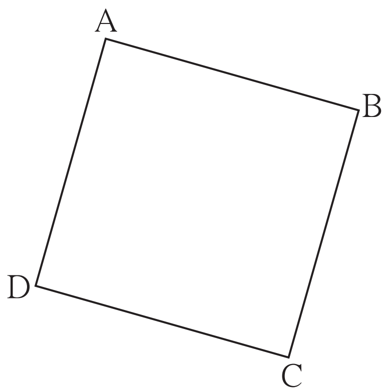


Fig. 5.9

$\therefore$ AB = BC = CD = DA and AC = BD

So, the lengths of four sides are equal and two diagonals are equal.

$\therefore$ (1,7), (4,2), (−1,−1) and (−4,4) are the vertices of a square.

Ex. (5) Find the co-ordinates of a point on Y- axis which is equidistant from M (−5,−2) and N(3,2)

Solution : Let point P(0, y) on Y- axis be equidistant from M(−5,−2) and N(3,2).

$$\therefore PM = PN \qquad \therefore PM^2 = PN^2$$

$$\therefore [0 - (-5)]^2 + [y - (-2)]^2 = (0 - 3)^2 + (y - 2)^2$$

$$\therefore 25 + (y + 2)^2 = 9 + y^2 - 4y + 4$$

$$\therefore 25 + y^2 + 4y + 4 = 13 + y^2 - 4y$$

$$\therefore 8y = -16 \quad \therefore y = -2$$

$\therefore$ the co-ordinates of the point on the Y-axis which is equidistant from M and N are M (0, −2).

Ex. (6) A(−3, −4), B(−5, 0), C(3, 0) are the vertices of Δ ABC. Find the co-ordinates of the circumcentre of Δ ABC.

Solution : Let, P(a, b) be the circumcentre of Δ ABC.

$\therefore$ point P is equidistant from A,B and C.

$\therefore PA^2 = PB^2 = PC^2$ .... (I) $\qquad \therefore PA^2 = PB^2$

$$(a + 3)^2 + (b + 4)^2 = (a + 5)^2 + (b - 0)^2$$

$$\therefore a^2 + 6a + 9 + b^2 + 8b + 16 = a^2 + 10a + 25 + b^2$$

$$\therefore -4a + 8b = 0$$

$$\therefore a - 2b = 0 \text{ .... (II)}$$

Similarly $PA^2 = PC^2$ .... (I) From

$$\therefore (a + 3)^2 + (b + 4)^2 = (a - 3)^2 + (b - 0)^2$$

$$\therefore a^2 + 6a + 9 + b^2 + 8b + 16 = a^2 - 6a + 9 + b^2$$

$$\therefore 12a + 8b = -16$$

$$\therefore 3a + 2b = -4 \text{ .... (III)}$$

Solving (II) and (III) we get $a = -1$, $b = -\frac{1}{2}$

$\therefore$ co-ordinates of circumcentre are $(-1, -\frac{1}{2})$.

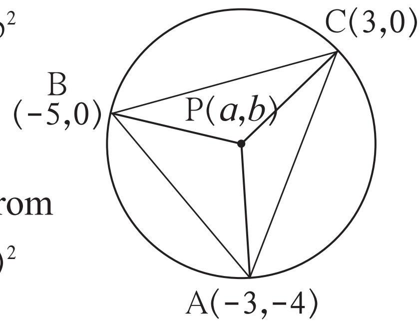


Fig. 5.10

Ex. (7) If point (x, y) is equidistant from points $(7, 1)$ and $(3, 5)$, show that $y = x-2$.

Solution : Let point P (x, y) be equidistant from points A(7, 1) and B(3, 5)

$$\therefore AP = BP$$
$$\therefore AP^2 = BP^2$$
$$\therefore (x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$
$$\therefore x^2 - 14x + 49 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 10y + 25$$
$$\therefore -8x + 8y = -16$$
$$\therefore x - y = 2$$
$$\therefore y = x - 2$$

Ex. (8) Find the value of y if distance between points A $(2, -2)$ and B $(-1, y)$ is 5.

Solution : $AB^2 = [(-1) - 2]^2 + [y - (-2)]^2$ .... by distance formula

$\therefore 5^2 = (-3)^2 + (y + 2)^2$

$\therefore 25 = 9 + (y + 2)^2$

$\therefore 16 = (y + 2)^2$

$\therefore y + 2 = \pm\sqrt{16}$

$\therefore y + 2 = \pm 4$

$\therefore y = 4 - 2$ or $y = -4 - 2$

$\therefore y = 2$ or $y = -6$

$\therefore$ value of y is 2 or -6.

Practice set 5.1

1. Find the distance between each of the following pairs of points.

   (1) A(2, 3), B(4, 1) (2) P(−5, 7), Q(−1, 3) (3) R(0, −3), S$(0, \frac{5}{2})$

   (4) L(5, −8), M(−7, −3) (5) T(−3, 6), R(9, −10) (6) W$(\frac{-7}{2}, 4)$, X(11, 4)

2. Determine whether the points are collinear.

   (1) A(1, −3), B(2, −5), C(−4, 7) (2) L(−2, 3), M(1, −3), N(5, 4)

   (3) R(0, 3), D(2, 1), S(3, −1) (4) P(−2, 3), Q(1, 2), R(4, 1)

3. Find the point on the X-axis which is equidistant from A(−3, 4) and B(1, −4).

4. Verify that points P(−2, 2), Q(2, 2) and R(2, 7) are vertices of a right angled triangle.

5. Show that points P(2, -2), Q(7, 3), R(11, -1) and S (6, -6) are vertices of a parallelogram.
6. Show that points A(-4, -7), B(-1, 2), C(8, 5) and D(5, -4) are vertices of a rhombus ABCD.
7. Find x if distance between points L(x, 7) and M(1, 15) is 10.
8. Show that the points A(1, 2), B(1, 6), C($1 + 2\sqrt{3}$, 4) are vertices of an equilateral triangle.

Let's recall.

Property of intercepts made by three parallel lines :

In the figure line l || line m || line n,

line p and line q are transversals,

then $\frac{AB}{BC} = \frac{DE}{EF}$

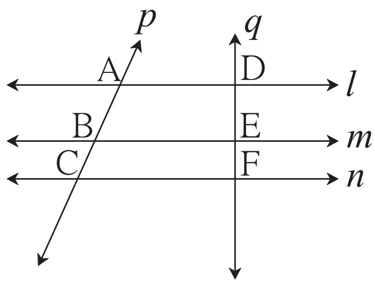


Fig. 5.11

Let's learn.

Division of a line segment

Fig. 5.12

In the figure, AP = 6 and PB = 10.

$\therefore \frac{AP}{PB} = \frac{6}{10} = \frac{3}{5}$

In other words it is said that 'point P divides the line segment AB in the ratio 3:5.

Let us see how to find the co-ordinates of a point on a segment which divides the segment in the given ratio.

Let's learn.

Section formula

In the figure 5.13, point P on the seg AB in XY plane, divides seg AB in the ratio $m : n$.

Assume $A(x_1, y_1)$ $B(x_2, y_2)$ and $P(x, y)$

Draw seg AC, seg PQ and seg BD perpendicular to X- axis.

$\therefore$ $C(x_1, 0)$; $Q(x, 0)$ and $D(x_2, 0)$.

$\therefore$ $\left.\begin{array}{l} CQ = x - x_1 \\ \text{and } QD = x_2 - x \end{array}\right\}$ .... (I)

seg AC || seg PQ || seg BD.

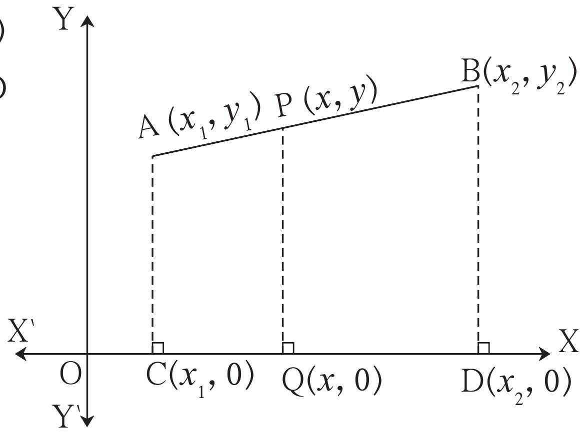


Fig. 5.13

$\therefore$ By the property of intercepts of three parallel lines, $\frac{AP}{PB} = \frac{CQ}{QD} = \frac{m}{n}$

Now $CQ = x - x_1$ and $QD = x_2 - x$ .... from (I)

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

$$\therefore n(x - x_1) = m(x_2 - x)$$

$$\therefore nx - nx_1 = mx_2 - mx$$

$$\therefore mx + nx = mx_2 + nx_1$$

$$\therefore x(m + n) = mx_2 + nx_1$$

$$\therefore x = \frac{mx_2 + nx_1}{m + n}$$

Similarly drawing perpendiculars from points A, P and B to Y- axis,

we get, $y = \frac{my_2 + ny_1}{m + n}$.

$\therefore$ co-ordinates of the point, which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$ are given by

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right).$$

Co-ordinates of the midpoint of a segment

If A(x_1, y_1)and B(x_2, y_2) are two points and P (x, y) is the midpoint of seg AB then $m = n$.

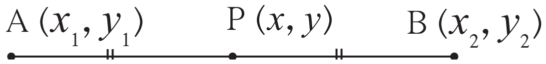


Fig. 5.14

$\therefore$ values of x and y can be written as

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$= \frac{mx_2 + mx_1}{m+m} \quad \because m = n$$

$$= \frac{m(x_1 + x_2)}{2m}$$

$$= \frac{x_1 + x_2}{2}$$

$$y = \frac{my_2 + ny_1}{m+n}$$

$$= \frac{my_2 + my_1}{m+m} \quad \because m = n$$

$$= \frac{m(y_1 + y_2)}{2m}$$

$$= \frac{y_1 + y_2}{2}$$

$\therefore$ co-ordinates of midpoint P are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

This is called as **midpoint formula**.

In the previous standard we have shown that $\frac{a+b}{2}$ is the midpoint of the segment joining two points indicating rational numbers a and b on a number line. Note that it is a special case of the above midpoint formula.

Solved Examples

Ex. (1) If A(3,5), B(7,9) and point Q divides seg AB in the ratio 2:3 then find co-ordinates of point Q.

Solution : In the given example let (x_1, y_1)= (3, 5)

and (x_2, y_2)= (7, 9) .

$m : n$ = 2:3

According to section formula,

$$x = \frac{mx_2 + nx_1}{m+n} = \frac{2\times7+3\times3}{2+3} = \frac{23}{5} \qquad y = \frac{my_2 + ny_1}{m+n} = \frac{2\times9+3\times5}{2+3} = \frac{33}{5}$$

$\therefore$ Co-ordinates of Q are $\left(\frac{23}{5}, \frac{33}{5}\right)$

Ex. (2) Find the co-ordinates of point P if P is the midpoint of a line segment AB with A(–4,2) and B(6,2).

Solution : In the given example, suppose

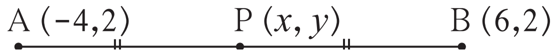


Fig. 5.15

$(-4, 2) = (x_1, y_1)$; $(6, 2) = (x_2, y_2)$ and co-ordinates of P are (x, y)

$\therefore$ according to midpoint theorem,

$$x = \frac{x_1 + x_2}{2} = \frac{-4+6}{2} = \frac{2}{2} = 1$$

$$y = \frac{y_1 + y_2}{2} = \frac{2+2}{2} = \frac{4}{2} = 2$$

$\therefore$ co-ordinates of midpoint P are (1,2) .

Let's recall.

We know that, medians of a triangle are concurrent .

The point of concurrence (centroid) divides the median in the ratio 2:1.

Let's learn.

Centroid formula

Suppose the co-ordinates of vertices of a triangle are given. Then we will find the co-ordinates of the centroid of the triangle.

In Δ ABC, $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices. Seg AD is a median and $G(x, y)$ is the centroid.

D is the mid point of line segment BC.

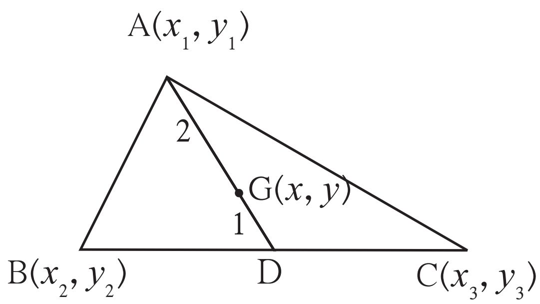


Fig. 5.16

$\therefore$ co-ordinates of point D are $x = \dfrac{x_2 + x_3}{2}$, $y = \dfrac{y_2 + y_3}{2}$ .... midpoint theorem

Point G(x, y) is centroid of triangle Δ ABC. $\therefore$ AG : GD = 2 : 1

$\therefore$ according to section formula,

$$x = \frac{2\left(\frac{x_2 + x_3}{2}\right) + 1 \times x_1}{2+1} = \frac{x_2 + x_3 + x_1}{3} = \frac{x_1 + x_2 + x_3}{3}$$

$$y = \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1 \times y_1}{2+1} = \frac{y_2 + y_3 + y_1}{3} = \frac{y_1 + y_2 + y_3}{3}$$

Thus if (x_1, y_1), (x_2, y_2) and $(x_3, y_3)$are the vertices of a triangle then the co-ordinates of the centroid are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$.

This is called the **centroid formula.**

Remember this!

- Section formula

  The co-ordinates of a point which divides the line segment joined by two distinct points (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$.

- Midpoint formula

  The co-ordinates of midpoint of a line segment joining two distinct points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

- Centroid formula

  If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle then co-ordinates of the centroid are $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$.

Solved Examples

Ex. (1) If point T divides the segment AB with A(-7,4) and B(-6,-5) in the ratio 7:2, find the co-ordinates of T.

Solution : Let the co-ordinates of T be (x, y).

$\therefore$ by the section formula,

$$x = \frac{mx_2 + nx_1}{m+n} = \frac{7\times(-6)+2\times(-7)}{7+2}$$

$$= \frac{-42-14}{9} = \frac{-56}{9}$$

$$y = \frac{my_2 + ny_1}{m+n} = \frac{7\times(-5)+2\times(4)}{7+2}$$

$$= \frac{-35+8}{9} = \frac{-27}{9} = -3$$

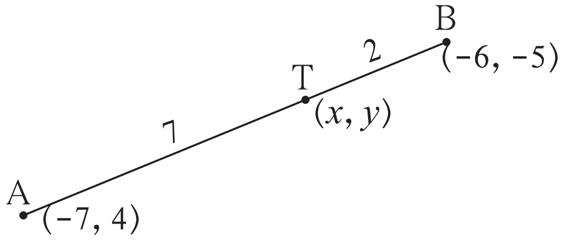


Fig. 5.17

$\therefore$ co-ordinates of point T are $\left(\frac{-56}{9}, -3\right)$.

Ex. (2) If point P(-4, 6) divides the line segment AB with A(-6, 10) and B(r, s) in the ratio 2:1, find the co-ordinates of B.

Solution : By section formula

$$-4 = \frac{2 \times r + 1 \times (-6)}{2 + 1}$$

$$\therefore \quad -4 = \frac{2r - 6}{3}$$

$$\therefore -12 = 2r - 6$$

$$\therefore \quad 2r = -6$$

$$\therefore \quad r = -3$$

$$6 = \frac{2 \times s + 1 \times 10}{2 + 1}$$

$$\therefore \quad 6 = \frac{2s + 10}{3}$$

$$\therefore 18 = 2s + 10$$

$$\therefore 2s = 8$$

$$\therefore \quad s = 4$$

$\therefore$ co-ordinates of point B are (-3, 4).

Ex. (3) A(15,5), B(9,20) and A-P-B. Find the ratio in which point P(11,15) divides segment AB.

Solution : Suppose, point P(11,15) divides segment AB in the ratio $m : n$

$\therefore$ by section formula,

$x = \frac{mx_2 + nx_1}{m+n}$

$\therefore\ 11 = \frac{9m+15n}{m+n}$

$\therefore\ 11m + 11n = 9m + 15n$

$\therefore\ 2m = 4n$

$\therefore\ \frac{m}{n} = \frac{4}{2} = \frac{2}{1}$

$\therefore$ The required ratio is 2 : 1.

Similarly, find the ratio using y co-ordinates. Write the conclusion.

Ex. (4) Find the co-ordinates of the points of trisection of the segment joining the points A (2,−2) and B(−7,4) .

(The two points that divide the line segment in three equal parts are called as points of trisection of the segment.)

Solution : Let points P and Q be the points of trisection of the line segment joining the points A and B.

Point P and Q divide line segment AB into three parts.

AP = PQ = QB .... (I)

$\frac{AP}{PB} = \frac{AP}{PQ+QB} = \frac{AP}{AP+AP} = \frac{AP}{2AP} = \frac{1}{2}$ .... From (I)

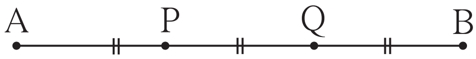


Fig. 5.18

Point P divides seg AB in the ratio 1:2.

x co-ordinate of point P = $\frac{1\times(-7)+2\times 2}{1+2} = \frac{-7+4}{3} = \frac{-3}{3} = -1$

y co-ordinate of point P = $\frac{1\times 4+2\times(-2)}{1+2} = \frac{4-4}{3} = \frac{0}{3} = 0$

Point Q divides seg AB in the ratio 2:1. $\therefore\ \frac{AQ}{QD} = \frac{2}{1}$

x co-ordinate of point Q = $\frac{2\times(-7)+1\times 2}{2+1} = \frac{-14+2}{3} = \frac{-12}{3} = -4$

y co-ordinate of point Q = $\frac{2\times 4+1\times -2}{2+1} = \frac{8-2}{3} = \frac{6}{3} = 2$

$\therefore$ co-ordinates of points of trisection are (−1, 0) and (−4, 2).

For more information :

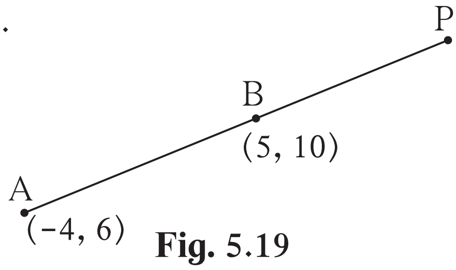
See how the external division of the line segment joining points A and B takes place.

Let us see how the co-ordinates of point P can be found out if P divides the line segment joining points A(−4, 6) and B(5, 10) in the ratio 3:1 externally.

$\frac{AP}{PB} = \frac{3}{1}$ that is AP is larger than PB and A−B−P.

$\frac{AP}{PB} = \frac{3}{1}$ that is $AP = 3k$, $BP = k$, then $AB = 2k$

$$\therefore \frac{AB}{BP} = \frac{2}{1}$$



Now point B divides seg AP in the ratio 2 : 1.

Fig. 5.19

We have learnt to find the coordinates of point P if co-ordinates of points A and B are known.

Practice set 5.2

1. Find the coordinates of point P if P divides the line segment joining the points A(−1,7) and B(4,−3) in the ratio 2 : 3.
2. In each of the following examples find the co-ordinates of point A which divides segment PQ in the ratio $a : b$.
   (1) P(−3, 7), Q(1, −4), $a : b = 2 : 1$
   (2) P(−2, −5), Q(4, 3), $a : b = 3 : 4$
   (3) P(2, 6), Q(−4, 1), $a : b = 1 : 2$
3. Find the ratio in which point T(−1, 6)divides the line segment joining the points P(−3, 10) and Q(6, −8).
4. Point P is the centre of the circle and AB is a diameter . Find the coordinates of point B if coordinates of point A and P are (2, −3) and (−2, 0) respectively.
5. Find the ratio in which point P(k, 7) divides the segment joining A(8, 9) and B(1, 2). Also find k .
6. Find the coordinates of midpoint of the segment joining the points (22, 20) and (0, 16).
7. Find the centroids of the triangles whose vertices are given below.
   (1)(−7, 6), (2, −2), (8, 5)
   (2) (3, −5), (4, 3), (11, −4)
   (3) (4, 7), (8, 4), (7, 11)

8. In Δ ABC, G (−4, −7) is the centroid. If A (−14, −19) and B(3, 5) then find the co-ordinates of C.

9. A(h, −6), B(2, 3) and C(−6, k) are the co-ordinates of vertices of a triangle whose centroid is G (1, 5). Find h and k.

10. Find the co-ordinates of the points of trisection of the line segment AB with A(2, 7) and B(−4, −8).

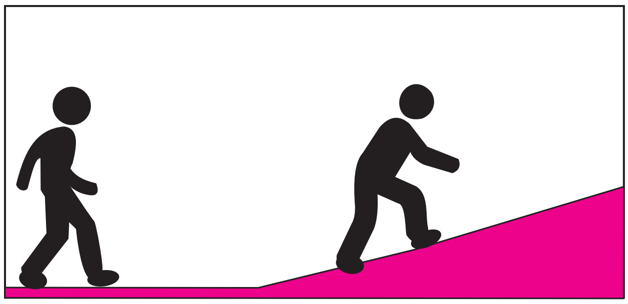
11. If A (−14, −10), B(6, −2) is given, find the coordinates of the points which divide segment AB into four equal parts.

12. If A (20, 10), B(0, 20) are given, find the coordinates of the points which divide segment AB into five congruent parts.

Let's learn.

Slope of a line

When we walk on a plane road we need not exert much effort but while climbing up a slope we need more effort. In science, we have studied that while climbing up a slope we have to work against gravitational force.



In co-ordinate geometry, slope of a line is an important concept. We will learn it through the following activity.

Activity I :

In the figure points A(−2, −5), B(0,−2), C(2,1), D(4,4), E(6,7) lie on line l. Observe the table which is made with the help of coordinates of these points on line l.

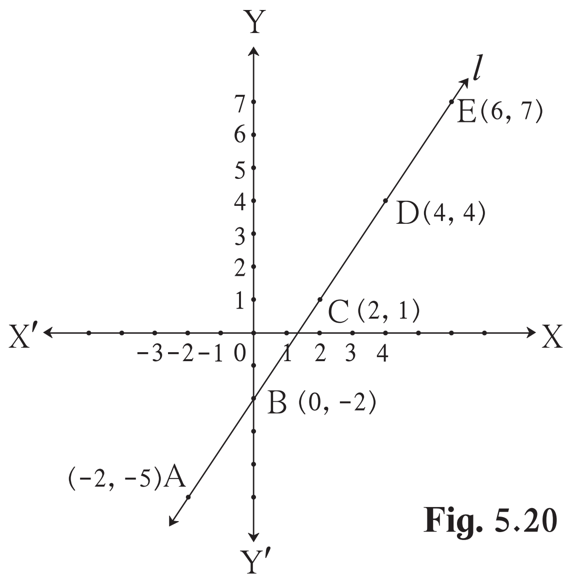


Fig. 5.20

Sr. No.	First point	Second point	Co-ordinates of first point (x_1, y_1)	Co-ordinates of second point (x_2, y_2)	$\frac{y_2 - y_1}{x_2 - x_1}$
1	C	E	(2, 1)	(6, 7)	$\frac{7-1}{6-2} = \frac{6}{4} = \frac{3}{2}$
2	A	D	(-2, -5)	(4, 4)	$\frac{4-(-5)}{4-(-2)} = \frac{9}{6} = \frac{3}{2}$
3	D	A	(4, 4)	(-2, -5)	$\frac{-5-4}{-2-4} = \frac{-9}{-6} = \frac{3}{2}$
4	B	C	--	--	--
5	C	A	--	--	--
6	A	C	--	--	--

Fill in the blank spaces in the above table. Similarly take some other pairs of points on line l and find the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ for each pair.

From this activity, we understand that for any two points (x_1, y_1) and (x_2, y_2) on line l, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is constant.

If (x_1, y_1) and (x_2, y_2) are any two points on line l, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is called the slope of the line l.

Generally slope is shown by letter m.

$$\therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Activity II : In the figure, some points on line l, t and n are given. Find the slopes of those lines.

Now you will know,

(1) Slopes of line l and line t are positive.

(2) Slope of line n is negative.

(3) Slope of line t is more than slope of line l.

(4) Slopes of lines l and t which make acute angles with X- axis, are positive.

(5) Slope of line n making obtuse angle with X- axis is negative.

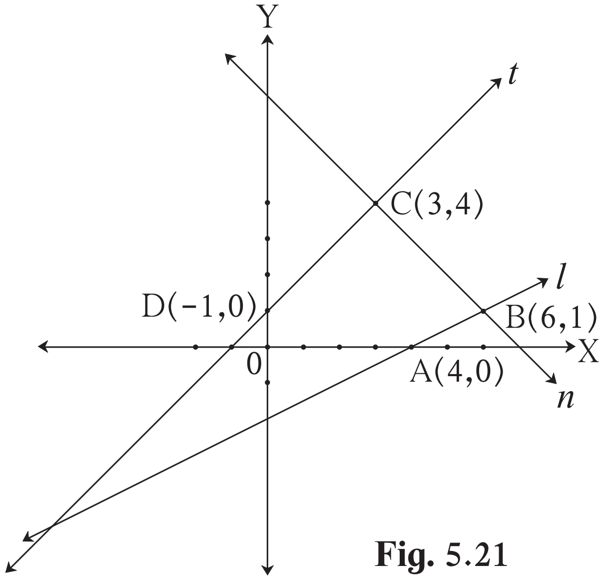


Fig. 5.21

Slopes of X-axis, Y-axis and lines parallel to axes.

In the figure 5.22, $(x_1, 0)$ and $(x_2, 0)$ are two points on the X- axis.

Slope of X- axis $= \dfrac{0 - 0}{x_2 - x_1} = 0$

In the same way $(0, y_1)$ and $(0, y_2)$ are two points on the Y- axis.

Slope of Y- axis $= \dfrac{y_2 - y_1}{0 - 0} = \dfrac{y_2 - y_1}{0}$,

But division by 0 is not possible.

$\therefore$ slope of Y- axis can not be determined.

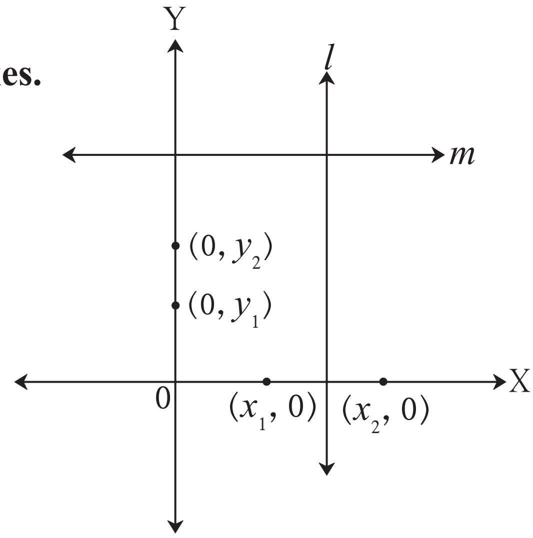


Fig. 5.22

Now try to find the slope of any line like line m which is parallel to X- axis. It will come out to be 0.

Similarly we cannot determine the slope of a line like line l which is parallel to Y- axis

Slope of line – using ratio in trigonometry

In the figure 5.23, points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on line l.

Line l intersects X axis in point T.

seg QS $\perp$ X- axis, seg PR $\perp$ seg QS $\therefore$ seg PR $\parallel$ seg TS .... corresponding angle test

$\therefore$ QR $= y_2 - y_1$ and PR $= x_2 - x_1$

$\therefore \quad \frac{QR}{PR} = \frac{y_2 - y_1}{x_2 - x_1}$ .... (I)

Line TQ makes an angle θ with the X- axis .

$\therefore \quad \frac{QR}{PR} = \tan\theta$ .... (II)

$\therefore$ From (I) and (II), $\frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$

$\therefore \quad m = \tan\theta$

Now seg PR || seg TS, line l is transversal

$\therefore \quad \angle QPR = \angle QTS$ ......corresponding angles

From this, we can define slope in another way. The tan ratio of an angle made by the line with the positive direction of X-axis is called the slope of that line.

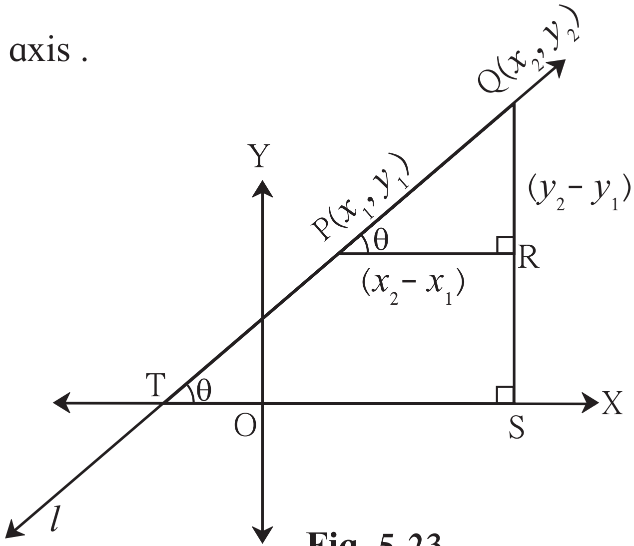


Fig. 5.23

When any two lines have same slope, these lines make equal angles with the positive direction of X- axis.

$\therefore$ These two lines are parallel.

Slope of Parallel Lines

Activity :

In the figure 5.22 both line l and line t make angle θ with the positive direction of X- axis.

$\therefore$ line l || line t .... corresponding angle test

Consider, point A(-3, 0) and point B(0, 3) on line l

Find the slope of line l.

Slope of line $l = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{\square - \square}{\square - \square} = \frac{\square}{\square}$

$= \square$

In the similar way, consider suitable points on the line t and find the slope of line t.

From this, you can verify that parallel lines have equal slopes.

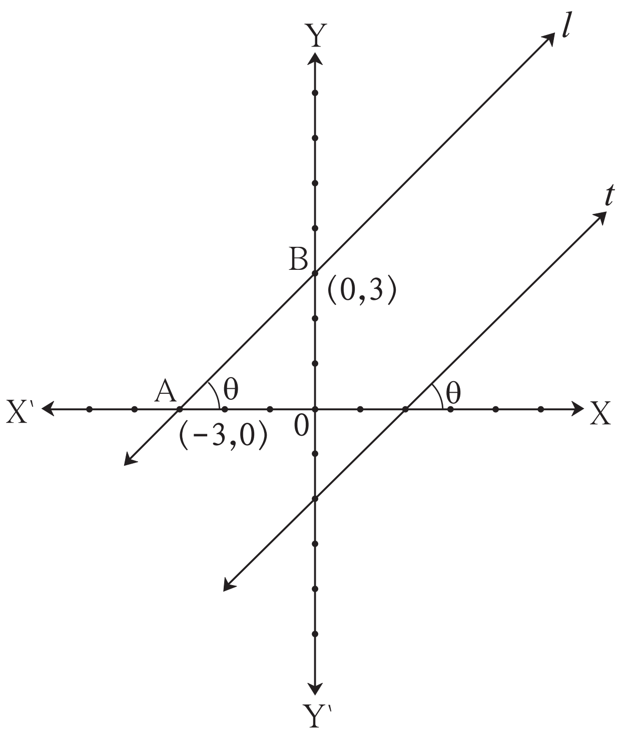


Fig. 5.24

Here $\theta = 45°$.

Use slope, $m = \tan\theta$ and verify that slopes of parallel lines are equal.

Similarly taking $\theta = 30°$, $\theta = 60°$ verify that slopes of parallel lines are equal.

Remember this!

The slope of X- axis and of any line parallel to X- axis is zero.

The slope of Y- axis and of any line parallel to Y- axis cannot be determined.

Solved Examples

EX. (1) Find the slope of the line passing through the points A $(-3, 5)$, and B $(4, -1)$

Solution : Let, $x_1 = -3$, $x_2 = 4$, $y_1 = 5$, $y_2 = -1$

$$\therefore \text{ Slope of line AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{4-(-3)} = \frac{-6}{7}$$

EX. (2) Show that points P$(-2, 3)$, Q$(1, 2)$, R$(4, 1)$ are collinear.

Solution : P$(-2, 3)$, Q$(1, 2)$ and R$(4, 1)$ are given points

$$\text{slope of line PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{1-(-2)} = -\frac{1}{3}$$

$$\text{Slope of line QR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{4-1} = -\frac{1}{3}$$

Slope of line PQ and line QR is equal.

But point Q lies on both the lines.

$\therefore$ Point P, Q, R are collinear.

EX. (3) If slope of the line joining points P$(k, 0)$ and Q$(-3, -2)$ is $\frac{2}{7}$ then find k.

Solution : P$(k, 0)$ and Q$(-3, -2)$

$$\text{Slope of line PQ} = \frac{-2-0}{-3-k} = \frac{-2}{-3-k}$$

But slope of line PQ is given to be $\frac{2}{7}$.

$$\therefore \frac{-2}{-3-k} = \frac{2}{7} \qquad \therefore k = 4$$

EX. (4) If A (6, 1), B (8, 2), C (9, 4) and D (7, 3) are the vertices of □ ABCD, show that □ ABCD is a parallelogram.

Solution : You know that Slope of line = $\frac{y_2 - y_1}{x_2 - x_1}$

Slope of line AB = $\frac{2-1}{8-6} = \frac{1}{2}$ .... (I)

Slope of line BC = $\frac{4-2}{9-8} = 2$ .... (II)

Slope of line CD = $\frac{3-4}{7-9} = \frac{1}{2}$ .... (III)

Slope of line DA = $\frac{3-1}{7-6} = 2$ .... (IV)

Slope of line AB = Slope of line CD .... From (I) and (III)

$\therefore$ line AB || line CD

Slope of line BC = Slope of line DA .... From (II) and (IV)

$\therefore$ line BC || line DA

Both the pairs of opposite sides of the quadrilateral are parallel

$\therefore$ □ ABCD is a parallelogram.

Practice set 5.3

1. Angles made by the line with the positive direction of X-axis are given. Find the slope of these lines.

(1) 45° (2) 60° (3) 90°

2. Find the slopes of the lines passing through the given points.

(1) A (2, 3) , B (4, 7) (2) P (-3, 1) , Q (5, -2)

(3) C (5, -2) , D (7, 3) (4) L (-2, -3) , M (-6, -8)

(5) E(-4, -2) , F (6, 3) (6) T (0, -3) , S (0, 4)

3. Determine whether the following points are collinear.

(1) A(-1, -1), B(0, 1), C(1, 3) (2) D(-2, -3), E(1, 0), F(2, 1)

(3) L(2, 5), M(3, 3), N(5, 1) (4) P(2, -5), Q(1, -3), R(-2, 3)

(5) R(1, -4), S(-2, 2), T(-3, 4) (6) A(-4, 4), K(-2, $\frac{5}{2}$), N(4, -2)

4. If A (1, -1),B (0, 4),C (-5, 3) are vertices of a triangle then find the slope of each side.

5. Show that A (-4, -7),B (-1, 2), C (8, 5) and D (5, -4) are the vertices of a parallelogram.

6. Find k, if R(1, −1), S (−2, k) and slope of line RS is −2.

7. Find k, if B(k, −5), C (1, 2) and slope of the line is 7.

8. Find k, if PQ || RS and P(2, 4), Q (3, 6), R(3, 1), S(5, k) .

Problem set 5

1. Fill in the blanks using correct alternatives.

(1) Seg AB is parallel to Y−axis and coordinates of point A are (1,3) then co-ordinates of point B can be ....

(A) (3,1) (B) (5,3) (C) (3,0) (D) (1,−3)

(2) Out of the following, point .... lies to the right of the origin on X− axis.

(A) (−2,0) (B) (0,2) (C) (2,3) (D) (2,0)

(3) Distance of point (−3,4) from the origin is ....

(A) 7 (B) 1 (C) 5 (D) −5

(4) A line makes an angle of 30° with the positive direction of X− axis. So the slope of the line is ....

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{3}$

2. Determine whether the given points are collinear.

(1) A(0,2) , B(1,−0.5), C(2,−3)

(2) P(1, 2) , Q(2, $\frac{8}{5}$) , R(3, $\frac{6}{5}$)

(3) L(1,2) , M(5,3) , N(8,6)

3. Find the coordinates of the midpoint of the line segment joining P(0,6) and Q(12,20).

4. Find the ratio in which the line segment joining the points A(3,8) and B(−9, 3) is divided by the Y− axis.

5. Find the point on X−axis which is equidistant from P(2,−5) and Q(−2,9).

6. Find the distances between the following points.

(i) A(a, 0), B(0, a) (ii) P(−6, −3), Q(−1, 9) (iii) R(−3a, a), S(a, −2a)

7. Find the coordinates of the circumcentre of a triangle whose vertices are (−3,1), (0,−2) and (1,3)

8. In the following examples, can the segment joining the given points form a triangle ? If triangle is formed, state the type of the triangle considering sides of the triangle.

(1) L(6,4) , M(–5,–3) , N(–6,8)

(2) P(–2,–6) , Q(–4,–2), R(–5,0)

(3) A($\sqrt{2}$, $\sqrt{2}$), B($-\sqrt{2}$, $-\sqrt{2}$), C($-\sqrt{6}$, $\sqrt{6}$)

9. Find k if the line passing through points P(–12,–3) and Q(4, k)has slope $\frac{1}{2}$.

10. Show that the line joining the points A(4, 8) and B(5, 5) is parallel to the line joining the points C(2,4) and D(1,7).

11. Show that points P(1,–2), Q(5,2), R(3,–1), S(–1,–5) are the vertices of a parallelogram

12. Show that the □ PQRS formed by P(2,1), Q(–1,3), R(–5,–3) and S(–2,–5) is a rectangle

13. Find the lengths of the medians of a triangle whose vertices are A(–1, 1), B(5, –3) and C(3, 5) .

14*. Find the coordinates of centroid of the triangles if points D(–7, 6), E(8, 5) and F(2, –2) are the mid points of the sides of that triangle.

15. Show that A(4, –1), B(6, 0), C(7, –2) and D(5, –3)are vertices of a square.

16. Find the coordinates of circumcentre and radius of circumcircle of Δ ABC if A(7, 1), B(3, 5) and C(2, 0) are given.

17. Given A(4,–3), B(8,5). Find the coordinates of the point that divides segment AB in the ratio 3:1.

18*. Find the type of the quadrilateral if points A(–4, –2), B(–3, –7) C(3, –2) and D(2, 3) are joined serially.

19*. The line segment AB is divided into five congruent parts at P, Q, R and S such that A–P–Q–R–S–B. If point Q(12, 14) and S(4, 18) are given find the coordinates of A, P, R,B.

20. Find the coordinates of the centre of the circle passing through the points P(6,–6), Q(3,–7)and R(3,3).

21*. Find the possible pairs of coordinates of the fourth vertex D of the parallelogram, if three of its vertices are A(5,6), B(1,–2)and C(3,–2).

22. Find the slope of the diagonals of a quadrilateral with vertices A(1,7), B(6,3), C(0,–3) and D(–3,3).

❑❑❑

PUX2MU

6 Trigonometry

Let's study.

- Trigonometric ratios
- Trigonometric identities
- Angle of elevation and angle of depression
- Problems based on heights and distances

Let's recall.

1. Fill in the blanks with reference to figure 6.1 .

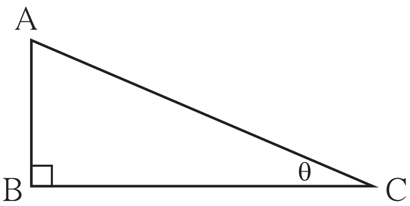


Fig. 6.1

$\sin\theta = \frac{\square}{\square}$, $\cos\theta = \frac{\square}{\square}$,

$\tan\theta = \frac{\square}{\square}$

2. Complete the relations in ratios given below .

(i) $\frac{\sin\theta}{\cos\theta} = \square$ (ii) $\sin\theta = \cos(90 - \square)$

(iii) $\cos\theta = \sin(90 - \square)$ (iv) $\tan\theta \times \tan(90 - \theta) = \square$

3. Complete the equation.

$\sin^2\theta + \cos^2\theta = \square$

4. Write the values of the following trigonometric ratios.

(i) $\sin 30^\circ = \frac{1}{\square}$ (ii) $\cos 30^\circ = \frac{\square}{\square}$ (iii) $\tan 30^\circ = \frac{\square}{\square}$

(iv) $\sin 60^\circ = \frac{\square}{\square}$ (v) $\cos 45^\circ = \frac{\square}{\square}$ (vi) $\tan 45^\circ = \square$

In std IX, we have studied some trigonometric ratios of some acute angles. Now we are going to study some more trigonometric ratios of acute angles.

Let's learn.

cosec, sec and cot ratios

Multiplicative inverse or the reciprocal of sine ratio is called cosecant ratio. It is written in brief as cosec. $\therefore$ $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

Similarly, multiplicative inverses or reciprocals of cosine and tangent ratios are called "secant" and "cotangent" ratios respectively. They are written in brief as sec and cot.

$\therefore$ $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

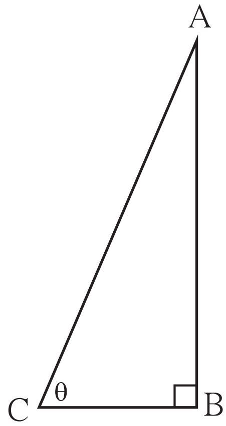


Fig. 6.2

In figure 6.2,

$$\sin\theta = \frac{AB}{AC}$$

$$\therefore \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{AB}{AC}} = \frac{AC}{AB}$$

It means,

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\tan\theta = \frac{AB}{BC}$$

$$\therefore \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{AB}{BC}}$$

$$\cot\theta = \frac{BC}{AB} = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\cos\theta = \frac{BC}{AC}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{BC}{AC}} = \frac{AC}{BC}$$

It means,

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

You know that,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

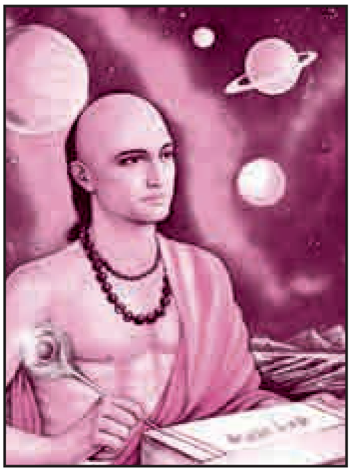
$$\therefore \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Remember this !

The relation between the trigonometric ratios, according to the definitions of cosec, sec and cot ratios

- $\frac{1}{\sin\theta} = \text{cosec}\ \theta \quad \therefore\ \sin\theta \times \text{cosec}\ \theta = 1$
- $\frac{1}{\cos\theta} = \sec\theta \quad \therefore\ \cos\theta \times \sec\theta = 1$
- $\frac{1}{\tan\theta} = \cot\theta \quad \therefore\ \tan\theta \times \cot\theta = 1$

For more information :



The great Indian mathematician Aryabhata was born in 476 A.D. in Kusumpur which was near Patna in Bihar. He has done important work in Arithmetic, Algebra and Geometry. In the book 'Aryabhatiya' he has written many mathematical formulae. For example,

(1) In an Arithmetic Progression, formulae for n^{th} term and the sum of first n terms.

(2) The formula to approximate $\sqrt{2}$

(3) The correct value of π upto four decimals, $\pi = 3.1416$.

In the study of Astronomy he used trigonometry and the sine ratio of an angle for the first time.

Comparing with the mathematics in the rest of the world at that time, his work was great and was studied all over India and was carried to Europe through Middle East.

Most observers at that time believed that the earth is immovable and the Sun, the Moon and stars move arround the earth. But Aryabhata noted that when we travel in a boat on the river, objects like trees, houses on the bank appear to move in the opposite direction. 'Similarly', he said 'the Sun, Moon and the stars are observed by people on the earth to be moving in the opposite direction while in reality the Earth moves !'

On 19 April 1975, India sent the first satellite in the space and it was named 'Aryabhata' to commemorate the great Mathematician of India.

✻ The table of the values of trigonometric ratios of angles 0°,30°,45°,60° and 90° .

Trigonometric ratio	Angle (θ)				
	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec θ $= \frac{1}{\sin \theta}$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ $= \frac{1}{\cos \theta}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot θ $= \frac{1}{\tan \theta}$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Let's learn.

Trigonometric identities

In the figure 6.3, Δ ABC is a right angled triangle, ∠B= 90°

(i) $\sin\theta = \frac{BC}{AC}$ (ii) $\cos\theta = \frac{AB}{AC}$

(iii) $\tan\theta = \frac{BC}{AB}$ (iv) $\text{cosec}\,\theta = \frac{AC}{BC}$

(v) $\sec\theta = \frac{AC}{AB}$ (vi) $\cot\theta = \frac{AB}{BC}$

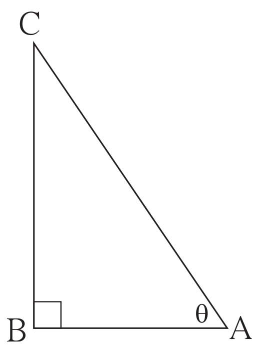


Fig. 6.3

By Pythagoras therom,

$BC^2 + AB^2 = AC^2$ . . . . .(I)

Dividing both the sides of (1) by AC^2

$$\frac{BC^2 + AB^2}{AC^2} = \frac{AC^2}{AC^2}$$

$\therefore \dfrac{BC^2}{AC^2} + \dfrac{AB^2}{AC^2} = 1$

$\therefore \left(\dfrac{BC}{AC}\right)^2 + \left(\dfrac{AB}{AC}\right)^2 = 1$

$\therefore (\sin\theta)^2 + (\cos\theta)^2 = 1$ .... [$(\sin\theta)^2$ is written as $\sin^2\theta$ and $(\cos\theta)^2$ is written as $\cos^2\theta$.]

$\sin^2\theta + \cos^2\theta = 1$ .... (II)

Now dividing both the sides of equation (II) by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ .... (III)

Dividing both the sides of equation (II) by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ .... (IV)

Relations (II),(III), and (IV) are fundamental trigonometric identities.

Solved Examples

Ex. (1) If $\sin\theta = \dfrac{20}{29}$ then find $\cos\theta$

Solution : **Method I**

We have

$\sin^2\theta + \cos^2\theta = 1$

$\left(\dfrac{20}{29}\right)^2 + \cos^2\theta = 1$

$\dfrac{400}{841} + \cos^2\theta = 1$

$\cos^2\theta = 1 - \dfrac{400}{841}$

$= \dfrac{441}{841}$

Taking square root of both sides.

$\cos\theta = \dfrac{21}{29}$

Method II

$\sin\theta = \dfrac{20}{29}$

from figure, $\sin\theta = \dfrac{AB}{AC}$

$\therefore$ AB = $20k$ and AC = $29k$

Let BC = x.

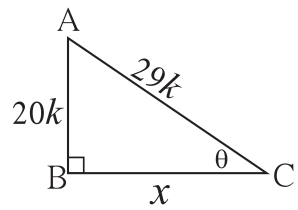


Fig. 6.4

According to Pythagoras therom,

$AB^2 + BC^2 = AC^2$

$(20k)^2 + x^2 = (29k)^2$

$400k^2 + x^2 = 841k^2$

$x^2 = 841k^2 - 400k^2$

$= 441k^2$

$\therefore \quad x = 21k$

$\therefore \cos\theta = \dfrac{BC}{AC} = \dfrac{21k}{29k} = \dfrac{21}{29}$

Ex. (2) If $\sec\theta = \frac{25}{7}$, find the value of $\tan\theta$.

Solution : Method I

we have ,

$1 + \tan^2\theta = \sec^2\theta$

$\therefore \quad 1 + \tan^2\theta = \left(\frac{25}{7}\right)^2$

$\therefore \quad \tan^2\theta = \frac{625}{49} - 1$

$= \frac{625 - 49}{49}$

$= \frac{576}{49}$

$\therefore \quad \tan\theta = \frac{24}{7}$

Method II

from the figure,

$\sec\theta = \frac{PR}{PQ}$

$\therefore \quad PQ = 7k,\ PR = 25k$

According to Pythagoras therom,

$PQ^2 + QR^2 = PR^2$

$\therefore \quad (7k)^2 + QR^2 = (25k)^2$

$\therefore \quad QR^2 = 625k^2 - 49k^2 = 576k^2$

$\therefore \quad QR = 24k$

Now, $\tan\theta = \frac{QR}{PQ} = \frac{24k}{7k} = \frac{24}{7}$

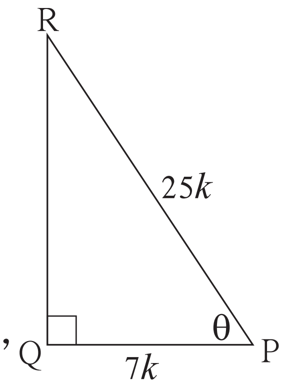


Fig. 6.5

Ex. (3) If $5\sin\theta - 12\cos\theta = 0$, find the values of $\sec\theta$ and $\mathrm{cosec}\theta$.

Solution : $5\sin\theta - 12\cos\theta = 0$

$\therefore \quad 5\sin\theta = 12\cos\theta$

$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{12}{5}$

$\therefore \quad \tan\theta = \frac{12}{5}$

we have,

$1 + \tan^2\theta = \sec^2\theta$

$\therefore \quad 1 + \left(\frac{12}{5}\right)^2 = \sec^2\theta$

$\therefore \quad 1 + \frac{144}{25} = \sec^2\theta$

$\therefore \quad \frac{25 + 144}{25} = \sec^2\theta$

$\therefore \quad \sec^2\theta = \frac{169}{25}$

$\therefore \quad \sec\theta = \frac{13}{5}$

$\therefore \quad \cos\theta = \frac{5}{13}$

Now, $\sin^2\theta + \cos^2\theta = 1$

$\therefore \quad \sin^2\theta = 1 - \cos^2\theta$

$\therefore \quad \sin^2\theta = 1 - \left(\frac{5}{13}\right)^2$

$= 1 - \frac{25}{169}$

$= \frac{144}{169}$

$\therefore \quad \sin\theta = \frac{12}{13}$

$\therefore \quad \mathrm{cosec}\theta = \frac{13}{12}$

Ex. (4) $\cos\theta = \frac{\sqrt{3}}{2}$ then find the value of $\frac{1-\sec\theta}{1+\operatorname{cosec}\theta}$.

Solution : Method I

$\cos\theta = \frac{\sqrt{3}}{2}$ $\therefore \sec\theta = \frac{2}{\sqrt{3}}$

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \sin^2\theta + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

$\therefore \sin^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore \sin\theta = \frac{1}{2}$ $\therefore \operatorname{cosec}\theta = 2$

$$\therefore \frac{1-\sec\theta}{1+\operatorname{cosec}\theta} = \frac{1-\frac{2}{\sqrt{3}}}{1+2}$$

$$= \frac{\frac{\sqrt{3}-2}{\sqrt{3}}}{3}$$

$$= \frac{\sqrt{3}-2}{3\sqrt{3}}$$

Method II

$\cos\theta = \frac{\sqrt{3}}{2}$

we know that, $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

$\therefore \theta = 30^\circ$

$\therefore \sec\theta = \sec 30^\circ = \frac{2}{\sqrt{3}}$

$\operatorname{cosec}\theta = \operatorname{cosec} 30^\circ = 2$

$$\therefore \frac{1-\sec\theta}{1+\operatorname{cosec}\theta} = \frac{1-\frac{2}{\sqrt{3}}}{1+2}$$

$$= \frac{\frac{\sqrt{3}-2}{\sqrt{3}}}{3}$$

$$= \frac{\sqrt{3}-2}{3\sqrt{3}}$$

Ex. (5) Show that $\sec x + \tan x = \sqrt{\frac{1+\sin x}{1-\sin x}}$

Solution : $\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$

$$= \frac{1+\sin x}{\cos x}$$

$$= \sqrt{\frac{(1+\sin x)^2}{\cos^2 x}}$$

$$= \sqrt{\frac{(1+\sin x)(1+\sin x)}{1-\sin^2 x}}$$

$$= \sqrt{\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}}$$

$$= \sqrt{\frac{1+\sin x}{1-\sin x}}$$

Ex. (6) Eliminate θ from given equations.

$x = a \cot\theta - b \operatorname{cosec}\theta$

$y = a \cot\theta + b \operatorname{cosec}\theta$

Solution : $x = a \cot\theta - b \operatorname{cosec}\theta$ .... (I)

$y = a \cot\theta + b \operatorname{cosec}\theta$ .... (II)

Adding equations (I) and (II).

$x + y = 2a \cot\theta$

$\therefore \cot\theta = \frac{x+y}{2a}$ .... (III)

Subtracting equation (II) from (I) ,

$y - x = 2b \operatorname{cosec}\theta$

$\therefore \operatorname{cosec}\theta = \frac{y-x}{2b}$ .... (IV)

Now, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\therefore \left(\frac{y-x}{2b}\right)^2 - \left(\frac{y+x}{2a}\right)^2 = 1$$

$$\therefore \frac{(y-x)^2}{4b^2} - \frac{(y+x)^2}{4a^2} = 1$$

or $\left(\frac{y-x}{b}\right)^2 - \left(\frac{y+x}{a}\right)^2 = 4$

Practice set 6.1

1. If $\sin\theta = \frac{7}{25}$, find the values of $\cos\theta$ and $\tan\theta$.

2. If $\tan\theta = \frac{3}{4}$, find the values of $\sec\theta$ and $\cos\theta$.

3. If $\cot\theta = \frac{40}{9}$, find the values of $\operatorname{cosec}\theta$ and $\sin\theta$.

4. If $5\sec\theta - 12\operatorname{cosec}\theta = 0$, find the values of $\sec\theta$, $\cos\theta$ and $\sin\theta$.

5. If $\tan\theta = 1$ then, find the values of $\frac{\sin\theta + \cos\theta}{\sec\theta + \operatorname{cosec}\theta}$.

6. Prove that:

(1) $\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$

(2) $\cos^2\theta(1 + \tan^2\theta) = 1$

(3) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

(4) $(\sec\theta - \cos\theta)(\cot\theta + \tan\theta) = \tan\theta\sec\theta$

(5) $\cot\theta + \tan\theta = \operatorname{cosec}\theta\sec\theta$

(6) $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

(7) $\sec^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$

(8) $\sec\theta + \tan\theta = \dfrac{\cos\theta}{1-\sin\theta}$

(9) If $\tan\theta + \dfrac{1}{\tan\theta} = 2$, then show that $\tan^2\theta + \dfrac{1}{\tan^2\theta} = 2$

(10) $\dfrac{\tan A}{(1+\tan^2 A)^2} + \dfrac{\cot A}{(1+\cot^2 A)^2} = \sin A\cos A$

(11) $\sec^4 A\,(1 - \sin^4 A) - 2\tan^2 A = 1$

(12) $\dfrac{\tan\theta}{\sec\theta - 1} = \dfrac{\tan\theta + \sec\theta + 1}{\tan\theta + \sec\theta - 1}$

Let's learn.

Application of trigonometry

Many times we need to know the height of a tower, building, tree or distance of a ship from the lighthouse or width of a river etc.

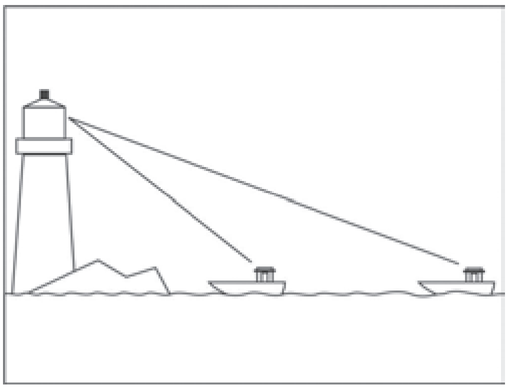


Fig. 6.6

We cannot measure them actually but we can find them with the help of trigonometric ratios.

For the purpose of computation, we draw a rough sketch to show the given information. 'Trees', 'hills' or 'towers' are vertical objects, so we shall represent them in the figure by segments which are perpendicular to the ground. We will not consider height of the observer and we shall normally regard observer's line of vision to be parallel to the horizontal level.

Let us study a few related terms.

(i) Line of vision : If the observer is standing at the location 'A',looking at an object 'B' then the line AB is called line of the vision.

(ii) Angle of elevation :

If an observer at A, observes the point B which is at a level higher than A and AM is the horizontal line, then $\angle$ BAM is called the angle of elevation.

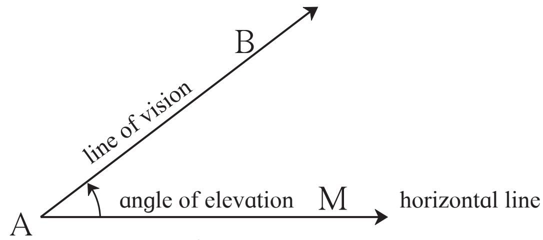


Fig. 6.7

(iii) Angle of depression :

If an observer at A, observes the point C which is at a level lower than A and AM is the horizontal line, the $\angle$ MAC is called the angle of depression.

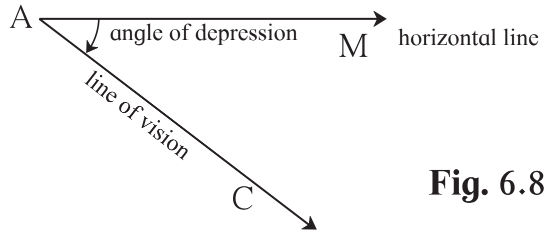


Fig. 6.8

When we see above the horizontal line, the angle formed is the angle of elevation. When we see below the horizontal line, the angle formed is the angle of depression.

Solved Examples

Ex. (1) An observer at a distance of 10 m from a tree looks at the top of the tree, the angle of elevation is 60°. What is the height of the tree ? ($\sqrt{3}$ = 1.73)

Solution : In figure 6.9, AB = h = height of the tree.

BC = 10 m, distance of the observer from the tree .

Angle of elevation (θ) = $\angle$ BCA = 60°

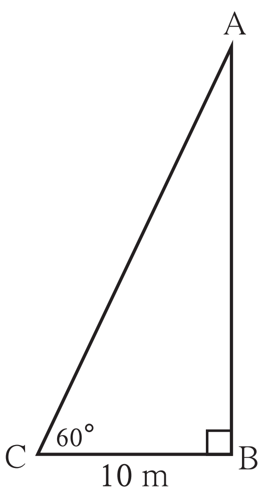


Fig. 6.9

from figure, $\tan\theta = \dfrac{AB}{BC}$ .... (I)

$\tan 60° = \sqrt{3}$ .... (II)

$\therefore \dfrac{AB}{BC} = \sqrt{3}$ .... from equation (I)and (II)

$\therefore$ AB = BC$\sqrt{3}$ = 10$\sqrt{3}$

$\therefore$ AB = 10 × 1.73 = 17.3 m

$\therefore$ height of the tree is 17.3m.

Ex. (2) From the top of a building, an observer is looking at a scooter parked at some distance away, makes an angle of depression of 30°. If the height of the building is 40m, find how far the scooter is from the building. ($\sqrt{3}$ = 1.73)

Solution: In the figure 6.10, AB is the building. A scooter is at C which is 'x' m away from the building.

In figure, 'A' is the position of the observer.

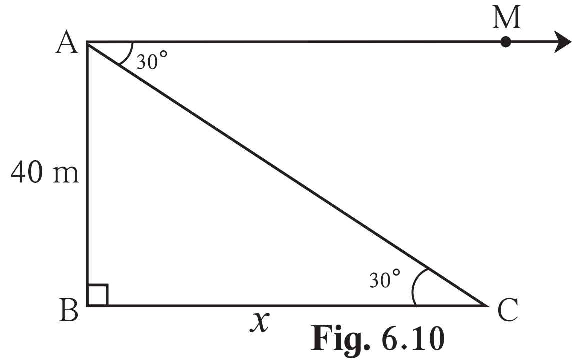


Fig. 6.10

AM is the horizontal line and ∠ MAC is the angle of depression.

∠ MAC and ∠ ACB are alternate angles.

from fig, $\tan 30° = \frac{AB}{BC}$

$\therefore \quad \frac{1}{\sqrt{3}} = \frac{40}{x}$

$\therefore \quad x = 40\sqrt{3}$

$= 40 \times 1.73$

$= 69.20$ m.

∴ the scooter is 69.20 m. away from the building.

Ex. (3) To find the width of the river, a man observes the top of a tower on the opposite bank making an angle of elevation of 61°. When he moves 50m backword from bank and observes the same top of the tower, his line of vision makes an angle of elevation of 35°. Find the height of the tower and width of the river. ($\tan 61° = 1.8$, $\tan 35° = 0.7$)

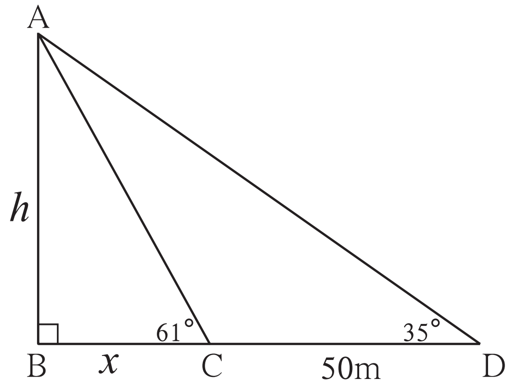


Fig. 6.11

Solution : seg AB shows the tower on the opposite bank. 'A' is the top of the tower and seg BC shows the width of the river. Let 'h' be the height of the tower and 'x' be the width of the river.

from figure, $\tan 61° = \frac{h}{x}$

$\therefore\ 1.8 = \dfrac{h}{x}$

$h = 1.8 \times x$

$10h = 18x$ .... (I)..... multipling by 10

In right angled Δ ABD,

$\tan 35 = \dfrac{h}{x + 50}$

$0.7 = \dfrac{h}{x + 50}$

$\therefore\ h = 0.7\,(x + 50)$

$\therefore\ 10h = 7\,(x + 50)$ .... (II)

$\therefore$ from equations (I) and (II) ,

$18x = 7(x + 50)$

$\therefore\ 18x = 7x + 350$

$\therefore\ 11x = 350$

$\therefore\ x = \dfrac{350}{11} = 31.82$

Now, $h = 1.8x = 1.8 \times 31.82$

$= 57.28$ m.

$\therefore$ width of the river = 31.82 m and height of tower = 57.28 m

Ex. (4) Roshani saw an eagle on the top of a tree at an angle of elevation of 61°, while she was standing at the door of her house. She went on the terrace of the house so that she could see it clearly. The terrace was at a height of 4m. While observing the eagle from there the angle of elevation was 52°. At what height from the ground was the eagle ? (Find the answer correct upto nearest integer)

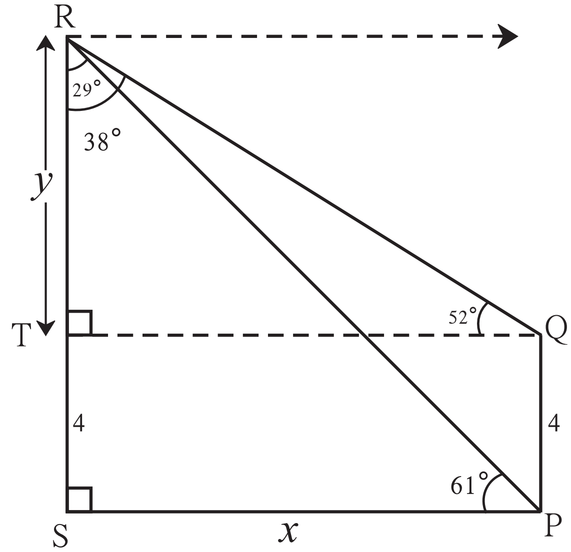


Fig. 6.12

(tan 61° = 1.80, tan 52° = 1.28, tan 29° = 0.55, tan 38° = 0.78)

Solution : In figure 6.12, PQ is the house and SR is the tree. The eagle is at R.

Draw seg QT $\perp$ seg RS.

$\therefore$ $\square$ TSPQ is a rectangle.

Let SP = x and TR = y

Now in Δ RSP, $\angle$ PRS = 90° − 61° = 29°

and in Δ RTQ, $\angle$ QRT = 90° − 52° = 38°

$\therefore \tan \angle PRS = \tan 29^\circ = \frac{SP}{RS}$

$\therefore 0.55 = \frac{x}{y+4}$

$\therefore x = 0.55(y + 4)$ .... (I)

Similarly, $\tan \angle QRT = \frac{TQ}{RT}$

$\therefore \tan 38^\circ = \frac{x}{y}$ .... [$\because$ SP = TQ = x]

$\therefore 0.78 = \frac{x}{y}$

$\therefore x = 0.78y$ .... (II)

$\therefore 0.78y = 0.55(y + 4)$ .... from (I) and (II)

$\therefore 78y = 55(y + 4)$

$\therefore 78y = 55y + 220$

$\therefore 23y = 220$

$\therefore y = 9.565 = 10$ (upto nearest integer)

$\therefore RS = y + 4 = 10 + 4 = 14$

$\therefore$ the eagle was at a height of 14 metre from the ground.

Ex. (5) A tree was broken due to storm. Its broken upper part was so inclined that its top touched the ground making an angle of 30°with the ground.The distance from the foot of the tree and the point where the top touched the ground was 10 metre. What was the height of the tree.

Solution: As shown in figure 6.13, suppose AB is the tree. It was broken at 'C' and its top touched at 'D'.

$\angle$ CDB = 30°, BD = 10 m, BC = x m

CA= CD = y m

In right angled Δ CDB,

$$\tan 30° = \frac{BC}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{x}{10}$$

$$x = \frac{10}{\sqrt{3}}$$

$$y = \frac{20}{\sqrt{3}}$$

$$x + y = \frac{10}{\sqrt{3}} + \frac{20}{\sqrt{3}}$$

$$= \frac{30}{\sqrt{3}}$$

$$x + y = 10\sqrt{3}$$

$\therefore$ height of the tree was $10\sqrt{3}$ m.

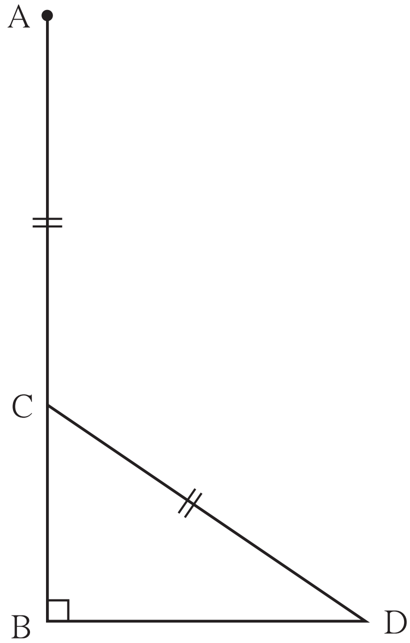


Fig. 6.13

Practice set 6.2

1. A person is standing at a distance of 80m from a church looking at its top. The angle of elevation is of 45°. Find the height of the church.
2. From the top of a lighthouse, an observer looking at a ship makes angle of depression of 60°. If the height of the lighthouse is 90 metre, then find how far the ship is from the lighthouse. ($\sqrt{3}$ =1.73)
3. Two buildings are facing each other on a road of width 12 metre. From the top of the first building, which is 10 metre high, the angle of elevation of the top of the second is found to be 60°. What is the height of the second building?
4. Two poles of heights 18 metre and 7 metre are erected on a ground. The length of the wire fastened at their tops in 22 metre. Find the angle made by the wire with the horizontal.
5. A storm broke a tree and the treetop rested 20 m from the base of the tree, making an angle of 60° with the horizontal. Find the height of the tree.
6. A kite is flying at a height of 60 m above the ground. The string attached to the kite is tied at the ground. It makes an angle of 60° with the ground. Assuming that the string is straight, find the length of the string. ($\sqrt{3}$ =1.73)

Problem set 6

1. Choose the correct alternative answer for the following questions.

(1) $\sin\theta \operatorname{cosec}\theta = ?$

(A) 1 (B) 0 (C) $\frac{1}{2}$ (D) $\sqrt{2}$

(2) $\operatorname{cosec}45^\circ = ?$

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{2}{\sqrt{3}}$

(3) $1 + \tan^2\theta = ?$

(A) $\cot^2\theta$ (B) $\operatorname{cosec}^2\theta$ (C) $\sec^2\theta$ (D) $\tan^2\theta$

(4) When we see at a higher level , from the horizontal line,angle formed is....... .

(A) angle of elevation. (B) angle of depression.

(C) 0 (D) straight angle.

2. If $\sin\theta = \frac{11}{61}$, find the values of $\cos\theta$ using trigonometric identity.

3. If $\tan\theta = 2$, find the values of other trigonometric ratios.

4. If $\sec\theta = \frac{13}{12}$, find the values of other trigonometric ratios.

5. Prove the following.

(1) $\sec\theta (1 - \sin\theta) (\sec\theta + \tan\theta) = 1$

(2) $(\sec\theta + \tan\theta) (1 - \sin\theta) = \cos\theta$

(3) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \times \operatorname{cosec}^2\theta$

(4) $\cot^2\theta - \tan^2\theta = \operatorname{cosec}^2\theta - \sec^2\theta$

(5) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

(6) $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$

(7) $\sec^6 x - \tan^6 x = 1 + 3\sec^2 x \times \tan^2 x$

(8) $\frac{\tan\theta}{\sec\theta+1} = \frac{\sec\theta-1}{\tan\theta}$

(9) $\frac{\tan^3\theta-1}{\tan\theta-1} = \sec^2\theta + \tan\theta$

(10) $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$

6. A boy standing at a distance of 48 meters from a building observes the top of the building and makes an angle of elevation of 30°.Find the height of the building.

7. From the top of the light house, an observer looks at a ship and finds the angle of depression to be 30°.If the height of the light-house is 100 meters, then find how far the ship is from the light-house.

8. Two buildings are in front of each other on a road of width 15 meters. From the top of the first building, having a height of 12 meter, the angle of elevation of the top of the second building is 30°.What is the height of the second building?

9. A ladder on the platform of a fire brigade van can be elevated at an angle of 70° to the maximum. The length of the ladder can be extended upto 20m. If the platform is 2m above the ground, find the maximum height from the ground upto which the ladder can reach. (sin 70° = 0.94)

10 *. While landing at an airport, a pilot made an angle of depression of 20°. Average speed of the plane was 200 km/hr. The plane reached the ground after 54 seconds. Find the height at which the plane was when it started landing. (sin 20° = 0.342)

❑❑❑

PV6XPH

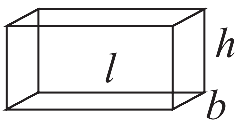
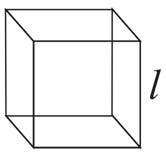
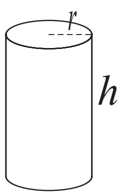
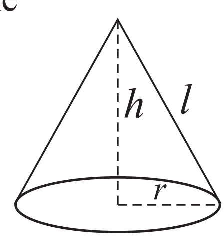
7 Mensuration

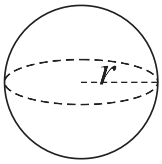
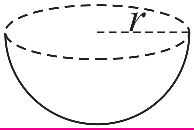
Let's study.

- Mixed examples on surface area and volume of different solid figures
- Arc of circle - length of arc
- Area of a sector
- Area of segment of a circle

Let's recall.

Last year we have studied surface area and volume of some three dimentional figures. Let us recall the formulae to find the surface areas and volumes.

No.	Three dimensional figure	Formulae
1 .	Cuboid 	Lateral surface area $= 2h\ (l + b)$ Total surface area $= 2\ (lb + bh + hl)$ Volume $= lbh$
2 .	Cube 	Lateral surface area $= 4l^2$ Total surface area $= 6l^2$ Volume $= l^3$
3 .	Cylinder 	Curved surface area $= 2\pi rh$ Total surface area $= 2\pi r\ (r + h)$ Volume $= \pi r^2h$
4 .	Cone 	Slant height $(l) = \sqrt{h^2 + r^2}$ Curved surface area $= \pi rl$ Total surface area $= \pi r\ (r + l)$ Volume $= \frac{1}{3} \times \pi r^2h$

No.	Three dimensional figure	Formulae
5.	Sphere 	Surface area = $4\pi r^2$ Volume = $\frac{4}{3}\pi r^3$
6.	Hemisphere 	Curved surface area = $2\pi r^2$ Total surface area of a solid hemisphere = $3\pi r^2$ Volume = $\frac{2}{3}\pi r^3$

Solve the following examples

Ex. (1)

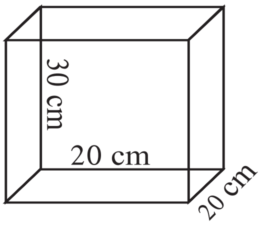


Fig 7.1

The length, breadth and height of an oil can are 20 cm, 20 cm and 30 cm respectively as shown in the adjacent figure.

How much oil will it contain ?

(1 litre = 1000 cm^3)

Ex. (2)

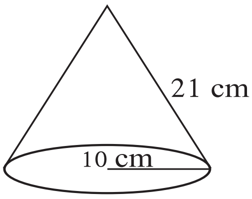


Fig 7.2

The adjoining figure shows the measures of a Joker's cap. How much cloth is needed to make such a cap ?

Let's think.

As shown in the adjacent figure, a sphere is placed in a cylinder. It touches the top, bottom and the curved surface of the cylinder. If radius of the base of the cylinder is 'r',

(1) What is the ratio of the radii of the sphere and the cylinder ?

(2) What is the ratio of the curved surface area of the cylinder and the surface area of the sphere ?

(3) What is the ratio of the volumes of the cylinder and the sphere ?

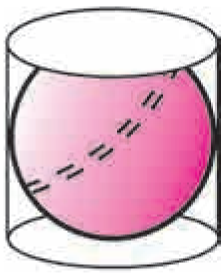


Fig. 7.3

Activity :

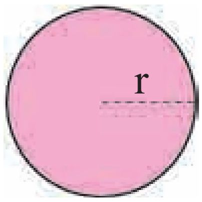


Fig. 7.4

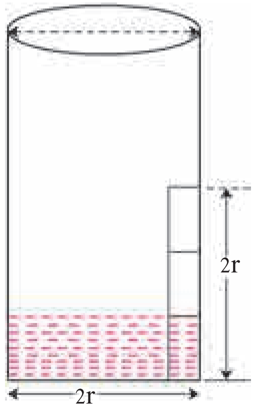


Fig. 7.5

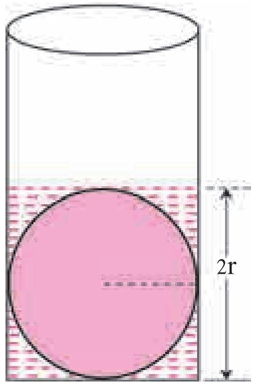


Fig. 7.6

As shown in the above figures, take a ball and a beaker of the same radius as that of the ball. Cut a strip of paper of length equal to the diameter of the beaker. Draw two lines on the strip dividing it into three equal parts. Stick it on the beaker straight up from the bottom. Fill water in the beaker upto the first mark of the strip from the bottom. Push the ball in the beaker slowly so that it touches its bottom. Observe how much the water level rises.

You will notice that the water level has risen exactly upto the total height of the strip. Try to understand how we get the formula for the volume of a the sphere. The shape of the beaker is cylindrical.

Therefore, the volume of the part of the beaker upto height 2r can be obtained by the formula of volume of a cylinder. Let us assume that the volume is V.

$$\therefore\ V = \pi r^2 h = \pi \times r^2 \times 2r = 2\pi r^3 \qquad (\because\ h = 2r)$$

But V = volume of the ball **+** volume of the water which was already in the beaker.

$$= \text{volume of the ball} + \frac{1}{3} \times 2\pi r^3$$

$$\therefore\ \text{volume of the ball} = V - \frac{1}{3} \times 2\pi r^3$$

$$= 2\pi r^3 - \frac{2}{3}\pi r^3$$

$$= \frac{6\pi r^3 - 2\pi r^3}{3} \qquad = \frac{4\pi r^3}{3}$$

Hence we get the formula of the volume of a sphere as $V = \frac{4}{3}\pi r^3$

(Now you can find the answer of the question number 3 relating to figure 7.3)

Solved Examples

Ex. (1) The radius and height of a cylindrical water reservoir is 2.8 m and 3.5 m respectively. How much maximum water can the tank hold ? A person needs 70 litre of water per day. For how many persons is the water sufficient for a day ? ($\pi = \frac{22}{7}$)

Solution : (r) = 2.8 m, (h) = 3.5 m, $\pi = \frac{22}{7}$

Capacity of the water reservoir = Volume of the cylindrical reservoir

$$= \pi r^2 h$$

$$= \frac{22}{7} \times 2.8 \times 2.8 \times 3.5$$

$$= 86.24 \text{ m}^3$$

$$= 86.24 \times 1000 \quad (\because 1m^3 = 1000 \text{ litre})$$

$$= 86240.00 \text{ litre.}$$

$\therefore$ the reservoir can hold 86240 litre of water.

The daily requirement of water of a person is 70 litre.

$\therefore$ water in the tank is sufficient for $\frac{86240}{70}$ = 1232 persons.

Ex. (2) How many solid cylinders of radius 10 cm and height 6 cm can be made by melting a solid sphere of radius 30 cm ?

Solution : Radius of a sphere, r = 30 cm

Radius of the cylinder, R = 10 cm

Height of the cylinder, H = 6 cm

Let the number of cylinders be n.

Volume of the sphere = n × volume of a cylinder

$$\therefore \; n = \frac{\text{Volume of the sphere}}{\text{Volume of a cylinder}}$$

$$= \frac{\frac{4}{3}\pi (r)^3}{\pi (R)^2 H}$$

$$= \frac{\frac{4}{3}\times(30)^3}{10^2 \times 6} = \frac{\frac{4}{3}\times 30\times 30\times 30}{10\times 10\times 6} = 60$$

$\therefore$ 60 cylinders can be made .

Ex. (3) A tent of a circus is such that its lower part is cylindrical and upper part is conical. The diameter of the base of the tent is 48 m and the height of the cylindrical part is 15 m. Total height of the tent is 33 m. Find area of canvas required to make the tent. Also find volume of air in the tent.

Solution : Total height of the tent = 33 m.

Let height of the cylindrical part be H

$\therefore$ H = 15 m.

Let the height of the conical part be h

$\therefore h = (33-15) = 18$ m.

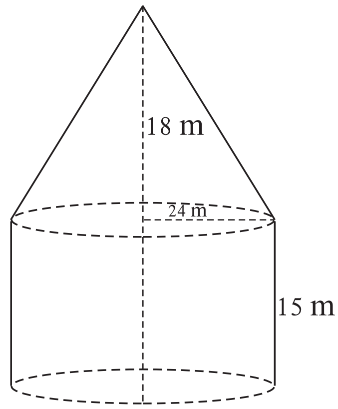


Fig 7.7

Slant height of cone, $l = \sqrt{r^2 + (h)^2}$

$= \sqrt{24^2 + 18^2}$

$= \sqrt{576 + 324}$

$= \sqrt{900}$

= 30 m.

Canvas required for tent = Curved surface area of the cylindrical part + Curved surface area of conical part

$= 2\pi rH + \pi rl$

$= \pi r\,(2H + l)$

$= \frac{22}{7} \times 24\,(2 \times 15 + 30)$

$= \frac{22}{7} \times 24 \times 60$

$= 4525.71\ m^2$

Volume of air in the tent = volume of cylinder + volume of cone

$= \pi r^2 H + \frac{1}{3}\pi r^2 h$

$= \pi r^2 \left(H + \frac{1}{3}h\right)$

$= \frac{22}{7} \times 24^2\,(15 + \frac{1}{3} \times 18)$

$= \frac{22}{7} \times 576 \times 21$

$= 38{,}016\ m^3$

$\therefore$ canvas required for the tent = $4525.71\ m^2$

$\therefore$ volume of air in the tent = $38{,}016\ m^3$.

Practice set 7.1

1. Find the volume of a cone if the radius of its base is 1.5 cm and its perpendicular height is 5 cm.

2. Find the volume of a sphere of diameter 6 cm.

3. Find the total surface area of a cylinder if the radius of its base is 5 cm and height is 40 cm.

4. Find the surface area of a sphere of radius 7 cm.

5. The dimensions of a cuboid are 44 cm, 21 cm, 12 cm. It is melted and a cone of height 24 cm is made. Find the radius of its base.

6.

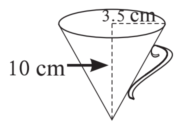


Fig 7.8
conical water jug

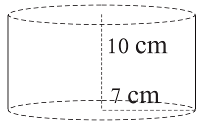


Fig 7.9
cylindrical water pot

Observe the measures of pots in figure 7.8 and 7.9. How many jugs of water can the cylindrical pot hold ?

7. A cylinder and a cone have equal bases. The height of the cylinder is 3 cm and the area of its base is 100 cm^2.The cone is placed upon the cylinder. Volume of the solid figure so formed is 500 cm^3. Find the total height of the figure.

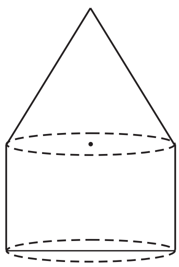


Fig 7.10

8. In figure 7.11, a toy made from a hemisphere, a cylinder and a cone is shown. Find the total area of the toy.

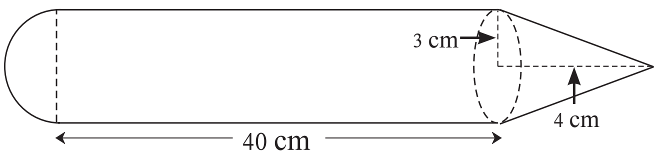


Fig. 7.11

9. In the figure 7.12, a cylindrical wrapper of flat tablets is shown. The radius of a tablet is 7 mm and its thickness is 5 mm. How many such tablets are wrapped in the wrapper ?

Fig. 7.12

10. Figure 7.13 shows a toy. Its lower part is a hemisphere and the upper part is a cone. Find the volume and the surface area of the toy from the measures shown in the figure.(π= 3.14)

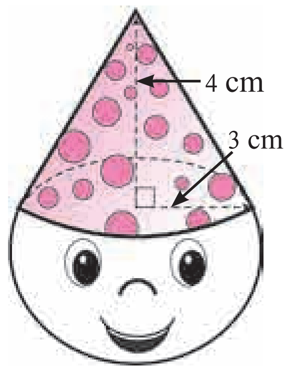


Fig. 7.13

11. Find the surface area and the volume of a beach ball shown in the figure.

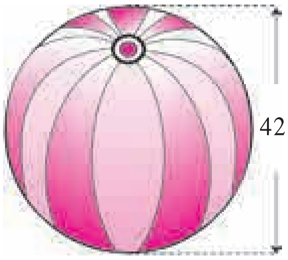


Fig. 7.14

12. As shown in the figure, a cylindrical glass contains water. A metal sphere of diameter 2 cm is immersed in it. Find the volume of the water.

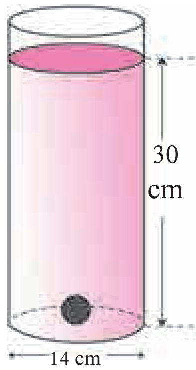


Fig. 7.15

Let's learn.

Frustum of a cone

The shape of glass used to drink water as well as the shape of water it contains, are examples of frustum of a cone.

Fig. 7.16

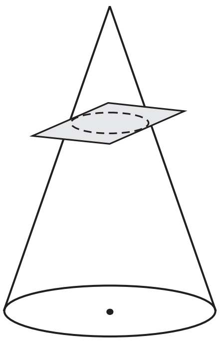


Fig. 7.17
A cone being cut

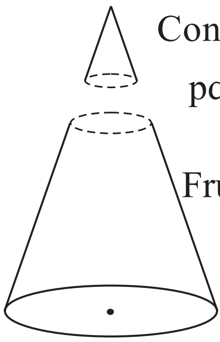


Fig. 7.18
Two parts of the cone

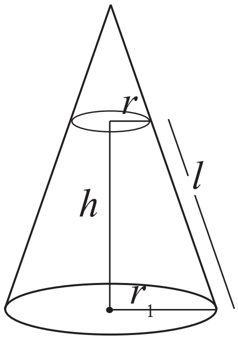


Fig. 7.19
Frustum

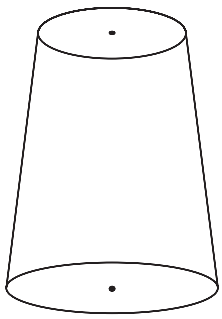


Fig. 7.20
A glass placed upside down

When a cone is cut parallel to its base we get two figures; one is a cone and the other is a frustum.

Volume and surface area of a frustum can be calculated by the formulae given below.

Remember this!

h = height of a frustum, l = slant height height of a frustum,

r_1 and r_2 = radii of circular faces of a frustum ($r_1 > r_2$)

Slant height of a frustum = $l = \sqrt{h^2 + (r_1 - r_2)^2}$

Curved surface area of a frustum = $\pi l\ (r_1 + r_2)$

Total surface area of a frustum = $\pi l\ (r_1 + r_2) + \pi r_1^2 + \pi r_2^2$

Volume of a frustum = $\frac{1}{3}\pi h\ (r_1^2 + r_2^2 + r_1 \times r_2)$

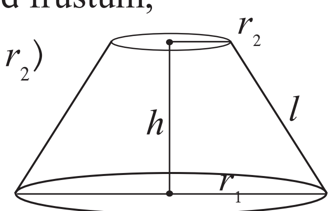


Fig. 7.21

Solved Examples

Ex. (1) A bucket is frustum shaped. Its height is 28 cm. Radii of circular faces are 12 cm and 15 cm. Find the capacity of the bucket. ($\pi = \frac{22}{7}$)

Solution : r_1 = 15 cm, r_2 = 12 cm, h = 28 cm

Capacity of the bucket = Volume of frustum

$$= \frac{1}{3}\pi h\ (r_1^2 + r_2^2 + r_1 \times r_2)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 28\ (15^2 \times 12^2 + 15 \times 12)$$

$$= \frac{22 \times 4}{3} \times (225 + 144 + 180)$$

$$= \frac{22 \times 4}{3} \times 549$$

$$= 88 \times 183$$

$$= 16104 \text{ cm}^3 = 16.104 \text{ litre}$$

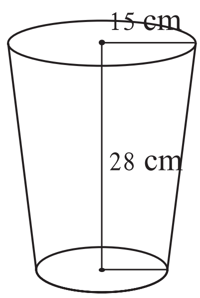


Fig. 7.22

$\therefore$ capacity of the bucket is 16.104 litre.

Ex. (2) Radii of the top and the base of a frustum are 14 cm, 8 cm respectively. Its height is 8 cm. Find its

i) curved surface area ii) total surface area iii) volume.

Solution : r_1 = 14 cm , r_2 = 8 cm, h = 8 cm

Slant height of the frustum = $l = \sqrt{h^2 + (r_1 - r_2)^2}$

$$= \sqrt{8^2 + (14 - 8)^2}$$

$$= \sqrt{64 + 36} = 10 \text{ cm}$$

Curved surface area of the frustum $= \pi(r_1 + r_2)\, l$

$= 3.14 \times (14 + 8) \times 10$

$= 690.8\ cm^2$

Total surface area of frustum $= \pi\, l\, (r_1 + r_2) + \pi r_1^2 + \pi r_2^2$

$= 3.14 \times 10\,(14 + 8) + 3.14 \times 14^2 + 3.14 \times 8^2$

$= 690.8 + 615.44 + 200.96$

$= 690.8 + 816.4$

$= 1507.2\ cm^2$

Volume of the frustum $= \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 \times r_2)$

$= \frac{1}{3} \times 3.14 \times 8\,(14^2 + 8^2 + 14 \times 8)$

$= 3114.88 cm^3$

Practice set 7.2

1. The radii of two circular ends of frustum shape bucket are 14 cm and 7 cm. Height of the bucket is 30 cm. How many liters of water it can hold ? (1 litre = 1000 cm^3)

2. The radii of ends of a frustum are 14 cm and 6 cm respectively and its height is 6 cm. Find its

i) curved surface area ii) total surface area. iii) volume ($\pi = 3.14$)

3. The circumferences of circular faces of a frustum are 132 cm and 88 cm and its height is 24 cm. To find the curved surface area of the frustum complete the following activity. ($\pi = \frac{22}{7}$).

circumference$_1$ $= 2\pi r_1 = 132$

$r_1 = \frac{132}{2\pi} = \square$

circumference$_2$ $= 2\pi r_2 = 88$

$r_2 = \frac{88}{2\pi} = \square$

slant height of frustum, $l = \sqrt{h^2 + (r_1 - r_2)^2}$

$= \sqrt{\square^2 + \square^2}$

$= \square$ cm

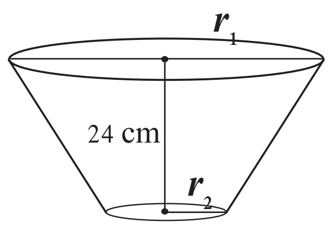


Fig. 7.23

curved surface area of the frustum = $\pi(r_1 + r_2)l$

$= \pi \times \square \times \square$

$= \square$ sq.cm.

Let's recall.

Complete the following table with the help of figure 7.24.

Type of arc	Name of the arc	Measure of the arc
Minor arc	arc AXB	....
....	arc AYB	....

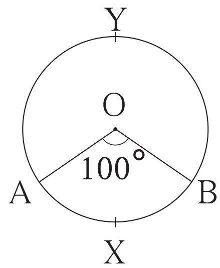


Fig. 7.24

Let's learn.

Sector of a circle

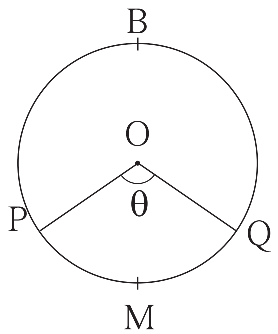


Fig. 7.25

In the adjacent figure, the central angle divides the circular region in two parts. Each of the parts is called a sector of the circle. Sector of a circle is the part enclosed by two radii of the circle and the arc joining their end points.

In the figure 7.25, O –PMQ and O–PBQ are two sectors of the circle.

Minor Sector :

Sector of a circle enclosed by two radii and their corresponding minor arc is called a 'minor sector'.

In the above figure O–PMQ is a minor sector.

Major Sector :

Sector of a circle that is enclosed by two radii and their corresponding major arc is called a 'major sector'.

In the above figure, O–PBQ is a major sector.

Area of a sector

Observe the figures below. Radii of all circles are equal. Observe the areas of the shaded regions and complete the following table.

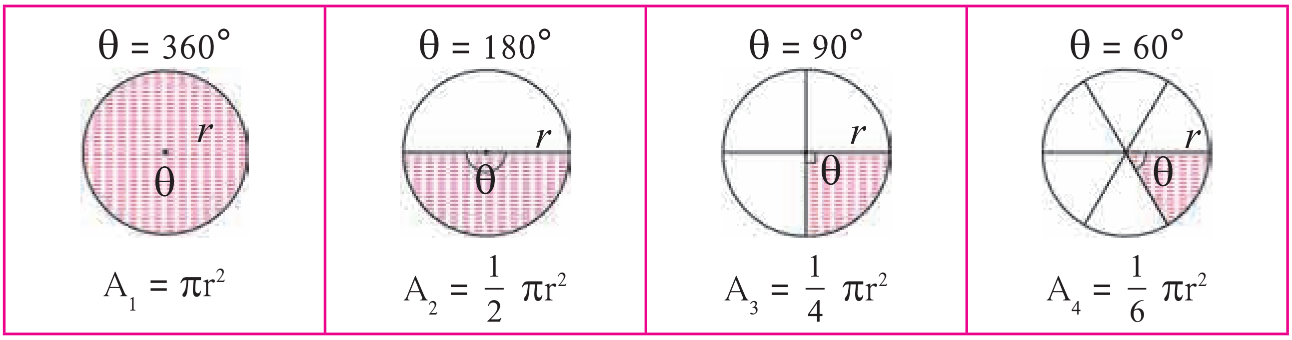


Fig. 7.26

Central angle of a circle is = 360° = complete angle

Central angle of a circle is = 360°, Area of a circle = πr^2			
Sector of circle	Measure of arc of the sector	$\frac{\theta}{360}$	Area of the sector A
A_1	360°	$\frac{360}{360} = 1$	$1 \times \pi r^2$
A_2	180°	$\frac{1}{2}$	$\frac{1}{2} \times \pi r^2$
A_3	90°	$\frac{1}{4}$	$\frac{1}{4} \times \pi r^2$
A_4	60°	....	....
A	θ	$\frac{\theta}{360}$	$\frac{\theta}{360} \times \pi r^2$

From the above table we see that, if measure of an arc of a circle is θ, then the area of its corresponding sector is obtained by multiplying area of the circle by $\frac{\theta}{360}$.

$$\text{Area of sector (A)} = \frac{\theta}{360} \times \pi r^2$$

From the formula,

$$\frac{A}{\pi r^2} = \frac{\theta}{360} \text{ ; that is } \frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\theta}{360}$$

Length of an arc

In the following figures, radii of all circles are equal. Observe the length of arc in each figure and complete the table.

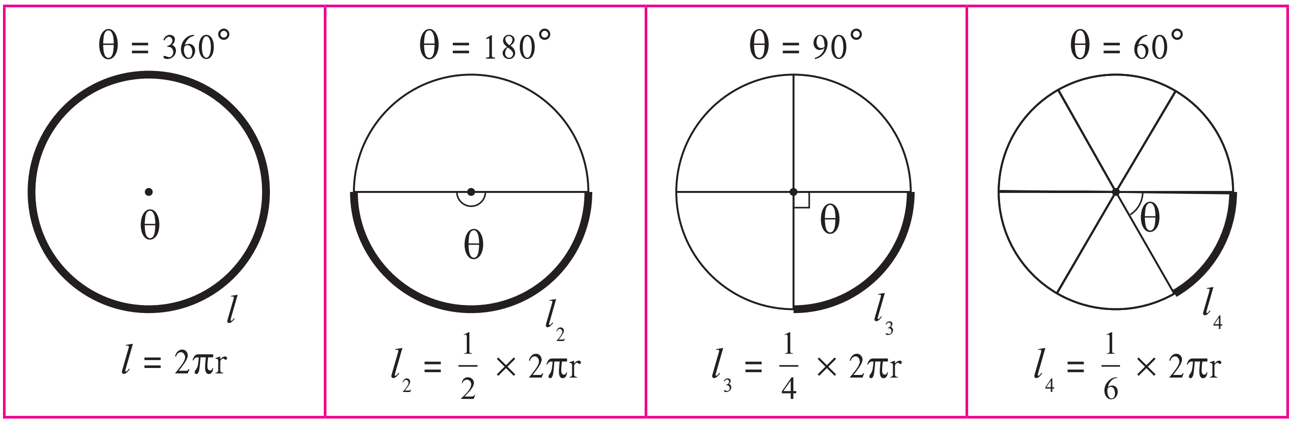


Fig. 7.27

Circumference of a circle = $2\pi r$			
Length of the arc	Measure of the arc (θ)	$\frac{\theta}{360}$	Length of the arc (l)
l_1	$360°$	$\frac{360}{360}=1$	$1 \times 2\pi r$
l_2	$180°$	$\frac{180}{360} = \frac{1}{2}$	$\frac{1}{2} \times 2\pi r$
l_3	$90°$	$\frac{90}{360} = \frac{1}{4}$	$\frac{1}{4} \times 2\pi r$
l_4	$60°$	....	....
l	θ	$\frac{\theta}{360}$	$\frac{\theta}{360} \times 2\pi r$

The pattern in the above table shows that, if measure of an arc of a circle is θ, then its length is obtained by multiplying the circumference of the circle by $\frac{\theta}{360}$.

$$\text{Length of an arc } (l) = \frac{\theta}{360} \times 2\pi r$$

From the formula, $\frac{l}{2\pi r} = \frac{\theta}{360}$

that is, $\frac{\text{Length of an arc}}{\text{Circumference}} = \frac{\theta}{360}$

A relation between length of an arc and area of the sector

Area of a sector, (A) = $\frac{\theta}{360} \times \pi r^2$ I

Length of an arc, (l) = $\frac{\theta}{360} \times 2\pi r$

$\therefore \frac{\theta}{360} = \frac{l}{2\pi r}$ II

$\therefore A = \frac{l}{2\pi r} \times \pi r^2$ ....From I and II

$A = \frac{1}{2} lr = \frac{lr}{2}$

$\therefore$ Area of a sector = $\frac{\text{Length of the arc} \times \text{Radius}}{2}$

Similarly, $\frac{A}{\pi r^2} = \frac{l}{2\pi r} = \frac{\theta}{360}$

Solved Examples

Ex. (1) The measure of a central angle of a circle is 150 $^\circ$ and radius of the circle is 21 cm. Find the length of the arc and area of the sector associated with the central angle.

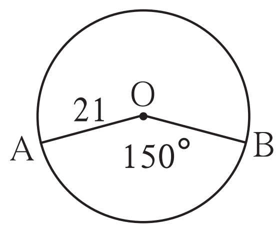


Fig. 7.28

Solution : r = 21 cm, θ = 150, π = $\frac{22}{7}$

Area of the sector, A = $\frac{\theta}{360} \times \pi r^2$

$= \frac{150}{360} \times \frac{22}{7} \times 21 \times 21$

$= \frac{1155}{2}$ = 577.5 cm^2

Length of the arc, l = $\frac{\theta}{360} \times 2\pi r$

$= \frac{150}{360} \times 2 \times \frac{22}{7} \times 21$

= 55 cm

ex. (2) In figure 7.29, P is the centre of the circle of radius 6 cm. Seg QR is a tangent at Q. If PR = 12, find the area of the shaded region. ($\sqrt{3}$ = 1.73)

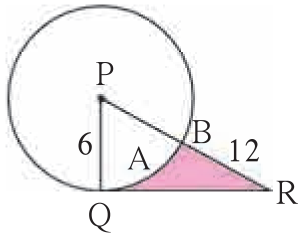


Fig. 7.29

Solution Radius joining point of contact of the tangent is perpendicular to the tangent.

$\therefore$ in Δ PQR, $\angle$ PQR = 90°, PQ = 6 cm, PR = 12 cm $\therefore$ PQ = $\frac{PR}{2}$

If one side of a right anglesd triangle is half the hypoteneus then angle opposite to, that side is of 30° measure

$\therefore \angle R = 30°$ and $\angle P = 60°$

$\therefore$ by 30°-60°-90° theorem, QR = $\frac{\sqrt{3}}{2} \times PR = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$

$\therefore$ QR = $6\sqrt{3}$ cm

$\therefore A(\Delta PQR) = \frac{1}{2}$ QR × PQ

$= \frac{1}{2} \times 6\sqrt{3} \times 6$

$= 18\sqrt{3} = 18 \times 1.73$

$= 31.14$ cm^2

Area of a sector = $\frac{\theta}{360} \times \pi r^2$

$A(P\text{-}QAB) = \frac{60}{360} \times 3.14 \times 6^2$

$= \frac{1}{6} \times 3.14 \times 6 \times 6 = 3.14 \times 6$

$= 18.84$ cm^2

Area of shaded region = A(PQR) − A(P-QAB)

$= 31.14 - 18.84$

$= 12.30$ cm^2

Area of the shaded region = 12.30 cm^2

Activity In figure 7.30, side of square ABCD is 7 cm. With centre D and radius DA, sector D − AXC is drawn. Fill in the following boxes properly and find out the area of the shaded region.

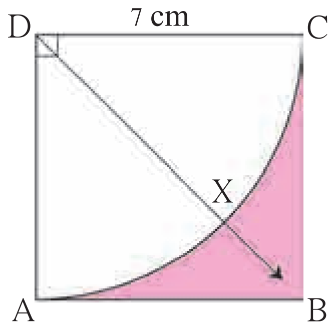


Fig. 7.30

Solution : Area of a square = ☐ (Formula)

= ☐

= 49 cm^2

Area of sector (D- AXC) = ☐ (Formula)

$= \frac{\square}{360} \times \frac{22}{7} \times \square$

= 38.5 cm^2

A (shaded region) = A ☐ − A ☐

= ☐ cm^2 − ☐ cm^2

= ☐ cm^2

Practice set 7.3

1. Radius of a circle is 10 cm. Measure of an arc of the crcleis 54°. Find the area of the sector associated with the arc. (π = 3.14)
2. Measure of an arc of a circle is 80 cm and its radius is 18 cm. Find the length of the arc. (π = 3.14)
3. Radius of a sector of a circle is 3.5 cm and length of its arc is 2.2 cm. Find the area of the sector.
4. Radius of a circle is 10 cm. Area of a sector of the sector is 100 cm^2. Find the area of its corresponding major sector. (π = 3.14)
5. Area of a sector of a circle of radius 15 cm is 30 cm^2. Find the length of the arc of the sector.
6. In the figure 7.31, radius of the circle is 7 cm and m(arc MBN) = 60°, find (1) Area of the circle .
   (2) A(O - MBN) .
   (3) A(O - MCN) .

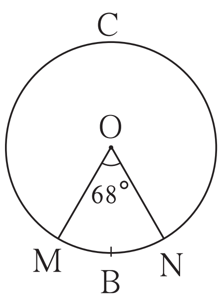


Fig. 7.31

7. In figure 7.32, radius of circle is 3.4 cm and perimeter of sector P-ABC is 12.8 cm. Find A(P-ABC).

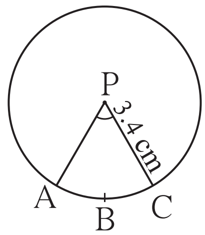


Fig. 7.32

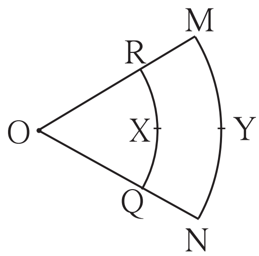


Fig. 7.33

8. In figure 7.33 O is the centre of the sector. $\angle$ ROQ = $\angle$ MON = 60°. OR = 7 cm, and OM = 21 cm. Find the lengths of arc RXQ and arc MYN. ($\pi = \frac{22}{7}$)

9. In figure 7.34, if A(P-ABC) = 154 cm^2 radius of the circle is 14 cm,

find (1) $\angle$ APC.

(2) l(arc ABC) .

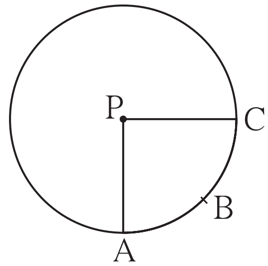


Fig. 7.34

10. Radius of a sector of a circle is 7 cm. If measure of arc of the sector is -

(1) 30° (2) 210° (3) three right angles;

find the area of the sector in each case.

11. The area of a minor sector of a circle is 3.85 cm^2 and the measure of its central angle is 36°. Find the radius of the circle.

12. In figure 7.35, □ PQRS is a rectangle. If PQ = 14 cm, QR = 21 cm, find the areas of the parts x, y and z .

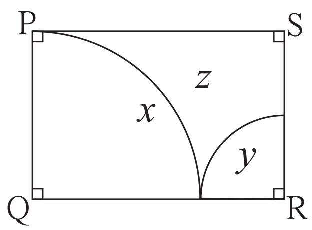


Fig. 7.35

13. Δ LMN is an equilateral triangle. LM = 14 cm. As shown in figure, three sectors are drawn with vertices as centres and radius7 cm. Find,

(1) A (Δ LMN)

(2) Area of any one of the sectors.

(3) Total area of all the three sectors.

(4) Area of the shaded region.

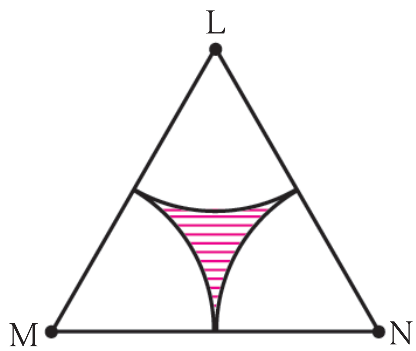


Fig. 7.36

Let's learn.

Segment of a circle

Segment of a circle is the region bounded by a chord and its corresponding arc of the circle.

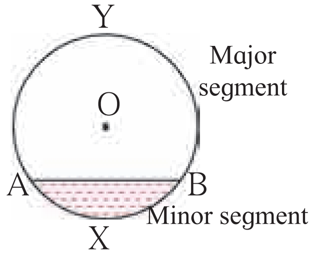


Fig. 7.37

Minor segment : The area enclosed by a chord and its corresponding minor arc is called a minor segment. In the figure, segment AXB is a minor segment.

Major segment : The area enclosed by a chord and its corresponding major arc is called a major segment. In the figure, seg AYB is a major segment.

Semicircular segment : A segment formed by a diameter of a circle is called a semicircular segment.

Area of a Segment

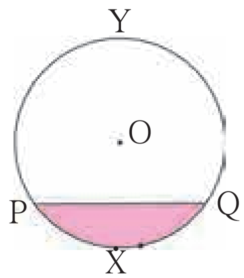


Fig. 7.38

In figure 7.38, PXQ is a minor segment and PYQ is a major segment.

How can we calculate the area of a minor segment?

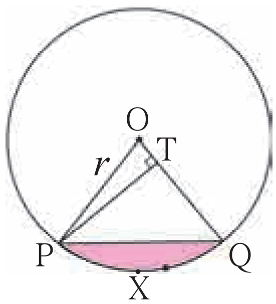


Fig. 7.39

In figure 7.39, draw radii OP and OQ. You know how to find the area of sector O−PXQ and Δ OPQ. We can get area of segment PXQ by substracting area of the triangle from the area of the sector.

A (segment PXQ) = A (O − PXQ) − A (Δ OPQ)

$$= \frac{\theta}{360} \times \pi r^2 - A(\Delta\ OPQ) \ldots\ldots\ldots\ldots \text{(I)}$$

In the figure, seg PT $\perp$ radius OQ.

Now, in Δ OTP $\sin\theta = \frac{PT}{OP}$

$\therefore$ PT = OP $\sin\theta$

$$PT = r \times \sin\theta \qquad (\because OP = r)$$

$$A(\Delta OPQ) = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times OQ \times PT$$

$$= \frac{1}{2} \times r \times r \sin\theta$$

$$= \frac{1}{2} \times r^2 \sin\theta \text{ .... (II)}$$

From (I) and (II) ,

$$A(\text{segment PXQ}) = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \times \sin\theta$$

$$= r^2 \left[\frac{\pi\theta}{360} - \frac{\sin\theta}{2} \right]$$

(Note that, we have studied the sine ratios of acute angles only. So we can use the above formula when $\theta \leq 90°$.)

Solved Examples

Ex. (1) In the figure 7.40, $\angle AOB = 30°$, $OA = 12$ cm . Find the area of the segment. ($\pi = 3.14$)

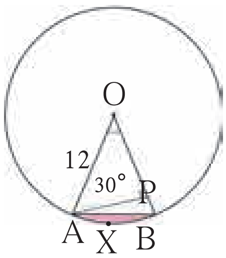


Fig. 7.40

Method I

$r = 12, \quad \theta = 30°, \quad \pi = 3.14$

$$A(O\text{–}AXB) = \frac{\theta}{360} \times \pi r^2$$

$$= \frac{30}{360} \times 3.14 \times 12^2$$

$$= 3.14 \times 12$$

$$= 37.68 \text{ cm}^2$$

$$A(\Delta OAB) = \frac{1}{2} r^2 \times \sin\theta$$

$$= \frac{1}{2} \times 12^2 \times \sin 30$$

$$= \frac{1}{2} \times 144 \times \frac{1}{2} \quad .....(\because \sin 30° = \frac{1}{2})$$

$$= 36 \text{ cm}^2$$

$$A(\text{segment } AXB) = A(O\text{-}AXB) - A(\Delta OAB)$$
$$= 37.68 - 36$$
$$= 1.68 \text{ cm}^2$$

Method II

$$A(\text{segment } AXB) = r^2\left[\frac{\pi\theta}{360} - \frac{\sin\theta}{2}\right]$$
$$= 12^2\left[\frac{3.14\times 30}{360} - \frac{\sin 30}{2}\right]$$
$$= 144\left[\frac{3.14}{12} - \frac{1}{2\times 2}\right]$$
$$= \frac{144}{4}\left[\frac{3.14}{3} - 1\right]$$
$$= 36\left[\frac{3.14-3}{3}\right]$$
$$= \frac{36}{3}\times 0.14$$
$$= 12 \times 0.14$$
$$= 1.68 \text{ cm}^2.$$

Ex. (2) The radius of a circle with centre P is 10 cm. If chord AB of the circle substends a right angle at P, find areas of the minor segment and the major segment. ($\pi = 3.14$)

Solution : $r = 10$ cm, $\theta = 90$, $\pi = 3.14$

$$A(P\text{-}AXB) = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{90}{360} \times 3.14 \times 10^2$$
$$= \frac{1}{4} \times 314$$
$$= 78.5 \text{ cm}^2$$

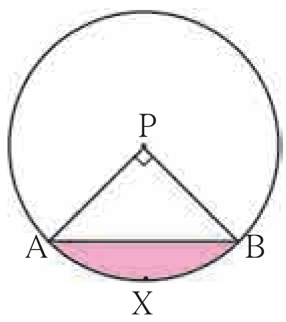


Fig. 7.41

$$A(\Delta APB) = \frac{1}{2} \text{ base} \times \text{height}$$
$$= \frac{1}{2} \times 10 \times 10$$
$$= 50 \text{ cm}^2$$

$$A(\text{minor segment}) = A(P\text{-}AXB) - A(\Delta PAB)$$
$$= 78.5 - 50$$
$$= 28.5 \text{ cm}^2$$

A(major segment) = A(circle) − A(minor segment)

$= 3.14 \times 10^2 - 28.5$

$= 314 - 28.5$

$= 285.5 \text{ cm}^2$

Ex. (3) A regular hexagon is inscribed in a circle of radius 14 cm. Find the area of the region between the circle and the hexagon. $(\pi = \frac{22}{7}, \sqrt{3} = 1.732)$

Solution : side of the hexagon = 14 cm

$$A(\text{hexagon}) = 6 \times \frac{\sqrt{3}}{4} \times (\text{side})^2$$

$$= 6 \times \frac{\sqrt{3}}{4} \times 14^2$$

$$= 509.208 \text{ cm}^2$$

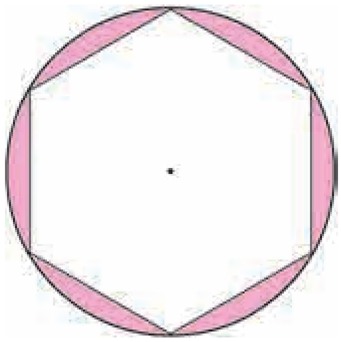


Fig. 7.42

$$A(\text{circle}) = \pi r^2$$

$$= \frac{22}{7} \times 14 \times 14$$

$$= 616 \text{ cm}^2$$

The area of the region between the circle and the hexagon

= A(circle) − A(hexagon)

$= 616 - 509.208$

$= 106.792 \text{ cm}^2$

Practice set 7.4

1. In figure 7.43, A is the centre of the circle. $\angle ABC = 45°$ and $AC = 7\sqrt{2}$ cm. Find the area of segment BXC.

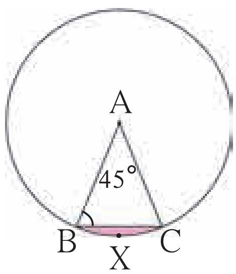


Fig. 7.43

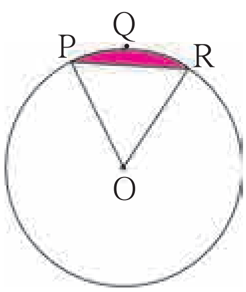


Fig. 7.44

2. In the figure 7.44, O is the centre of the circle. $m(\text{arc PQR}) = 60°$ OP = 10 cm.
Find the area of the shaded region.
$(\pi = 3.14, \sqrt{3} = 1.73)$

3. In the figure 7.45, if A is the centre of the circle. ∠ PAR = 30°, AP = 7.5, find the area of the segment PQR

(π = 3.14)

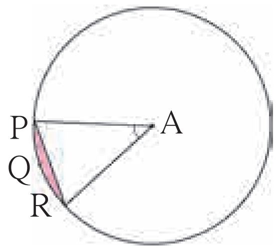


Fig. 7.45

4.

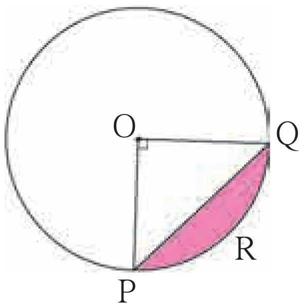


Fig. 7.46

In the figure 7.46, if O is the centre of the circle, PQ is a chord. ∠ POQ = 90°, area of shaded region is 114 cm^2, find the radius of the circle. (π = 3.14)

5. A chord PQ of a circle with radius 15 cm subtends an angle of 60° with the centre of the circle. Find the area of the minor as well as the major segment.

($\pi = 3.14$, $\sqrt{3} = 1.73$)

Problem set 7

1. Choose the correct alternative answer for each of the following questions.

(1) The ratio of circumference and area of a circle is 2:7. Find its circumference.

(A) 14π (B) $\frac{7}{\pi}$ (C) 7π (D) $\frac{14}{\pi}$

(2) If measure of an arc of a circle is 160° and its length is 44 cm, find the circumference of the circle.

(A) 66 cm (B) 44 cm (C) 160 cm (D) 99 cm

(3) Find the perimeter of a sector of a circle if its measure is 90 ° and radius is 7 cm.

(A) 44 cm (B) 25 cm (C) 36 cm (D) 56 cm

(4) Find the curved surface area of a cone of radius 7 cm and height 24 cm.

(A) 440 cm^2 (B) 550 cm^2 (C) 330 cm^2 (D) 110 cm^2

(5) The curved surface area of a cylinder is 440 cm^2 and its radius is 5 cm. Find its height.

(A) $\frac{44}{\pi}$ cm (B) 22π cm (C) 44π cm (D) $\frac{22}{\pi}$ cm

(6) A cone was melted and cast into a cylinder of the same radius as that of the base of the cone. If the height of the cylinder is 5 cm, find the height of the cone.

(A) 15 cm (B) 10 cm (C) 18 cm (D) 5 cm

(7) Find the volume of a cube of side 0.01 cm.

(A) 1 cm^3 (B) 0.001 cm^3 (C) 0.0001 cm^3 (D) 0.000001 cm^3

(8) Find the side of a cube of volume 1 m^3.

(A) 1 cm (B) 10 cm (C) 100 cm (D)1000 cm

2. A washing tub in the shape of a frustum of a cone has height 21 cm. The radii of the circular top and bottom are 20 cm and 15 cm respectively. What is the capacity of the tub ? ($\pi = \frac{22}{7}$)

3 ★. Some plastic balls of radius 1 cm were melted and cast into a tube. The thickness, length and outer radius of the tube were 2 cm, 90 cm and 30 cm respectively. How many balls were melted to make the tube ?

4. A metal parallelopiped of measures 16 cm × 11cm × 10 cm was melted to make coins. How many coins were made if the thickness and diameter of each coin was 2 mm and 2 cm respectively ?

5. The diameter and length of a roller is 120 cm and 84 cm respectively. To level the ground, 200 rotations of the roller are required. Find the expenditure to level the ground at the rate of Rs. 10 per sq.m.

6. The diameter and thickness of a hollow metals sphere are 12 cm and 0.01 m respectively. The density of the metal is 8.88 gm per cm^3. Find the outer surface area and mass of the sphere.

7. A cylindrical bucket of diameter 28 cm and height 20 cm was full of sand. When the sand in the bucket was poured on the ground, the sand got converted into a shape of a cone. If the height of the cone was 14 cm, what was the base area of the cone ?

8. The radius of a metallic sphere is 9 cm. It was melted to make a wire of diameter 4 mm. Find the length of the wire.

9. The area of a sector of a circle of 6 cm radius is 15π sq.cm. Find the measure of the arc and length of the arc corresponding to the sector.

10.

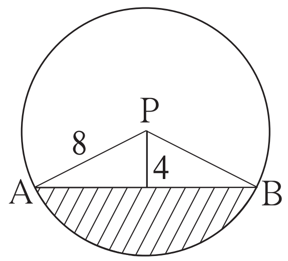


Fig.7.47

In the figure 7.47, seg AB is a chord of a circle with centre P. If PA = 8 cm and distance of chord AB from the centre P is 4 cm, find the area of the shaded portion. ($\pi = 3.14$, $\sqrt{3} = 1.73$)

11. In the figure 7.48, square ABCD is inscribed in the sector A-PCQ. The radius of sector C-BXD is 20 cm. Complete the following activity to find the area of shaded region.

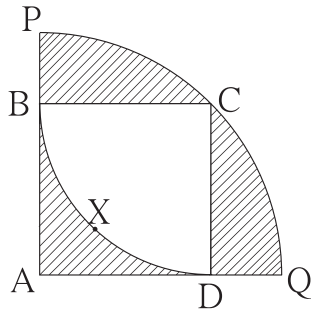


Fig. 7.48

Solution : Side of square ABCD = radius of sector C-BXD = $\square$ cm

Area of square = $(\text{side})^2 = \square^2 = \square$ ..... (I)

Area of shaded region inside the square

= Area of square ABCD − Area of sector C-BXD

$$= \square - \frac{\theta}{360} \times \pi r^2$$

$$= \square - \frac{90}{360} \times \frac{3.14}{1} \times \frac{400}{1}$$

$$= \square - 314$$

$$= \square$$

Radius of bigger sector = Length of diagonal of square ABCD

$= 20\sqrt{2}$

Area of the shaded regions outside the square

= Area of sector A-PCQ − Area of square ABCD

= A(A-PCQ) − A($\square$ABCD)

$$= \left(\frac{\theta}{360} \times \pi \times r^2\right) - \square^2$$

$$= \frac{90}{360} \times 3.14\ (20\sqrt{2})^2 - (20)^2$$

$$= \square - \square$$

$$= \square$$

$\therefore$ total area of the shaded region = 86 + 228 = 314 sq.cm.

12.

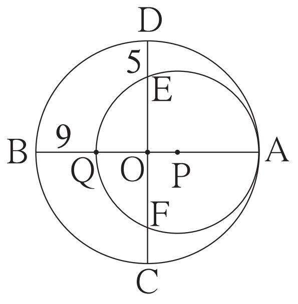


Fig. 7.49

In the figure 7.49, two circles with centres O and P are touching internally at point A. If BQ = 9, DE = 5, complete the following activity to find the radii of the circles.

Solution : Let the radius of the bigger circle be R and that of smaller circle be r.

OA, OB, OC and OD are the radii of the bigger circle

$\therefore$ OA = OB = OC = OD = R

PQ = PA = r

OQ = OB − BQ = $\square$

OE = OD − DE = $\square$

As the chords QA and EF of the circle with centre P intersect in the interior of the circle, so by the property of internal division of two chords of a circle,

$OQ \times OA = OE \times OF$

$\square \times R = \square \times \square$ .... ($\because$ OE = OF)

$R^2 - 9R = R^2 - 10R + 25$

R = $\square$

AQ = 2r = AB − BQ

2r = 50 − 9 = 41

r = $\square$ = $\square$

PVFTR5

ANSWERS

Chapter 1 Similarity

Practice set 1.1

1. $\frac{3}{4}$ **2.** $\frac{1}{2}$ **3.** 3 **4.** 1:1 **5.** (1) $\frac{BQ}{BC}$, (2) $\frac{PQ}{AD}$, (3) $\frac{BC}{DC}$, (4) $\frac{DC \times AD}{QC \times PQ}$

Practice set 1.2

1. (1) is a bisector. (2) is not a bisector. (3) is a bisector.

2. $\frac{PN}{NR} = \frac{PM}{MQ} = \frac{3}{2}$, therefore line NM || side RQ **3.** QP = 3.5 **5.** BQ = 17.5

6. QP = 22.4 **7.** x = 6 ; AE = 18 **8.** LT = 4.8 **9.** x = 10

10. Given, XQ, PD, Given, $\frac{\boxed{XR}}{\boxed{RF}} = \frac{\boxed{XQ}}{\boxed{QE}}$, Basic propotionality theorem, $\frac{\boxed{XP}}{\boxed{PD}} = \frac{\boxed{XR}}{\boxed{RF}}$

Practice set 1.3

1. Δ ABC ~ Δ EDC, AA test **2.** Δ PQR ~ Δ LMN ; SSS test of similarity

3. 12 metre **4.** AC = 10.5 **6.** OD = 4.5

Practice set 1.4

1. Ratio of areas = 9 : 25 **2.** $\boxed{PQ^2}$, $\frac{\boxed{4}}{\boxed{9}}$ **3.** $\boxed{A(\Delta PQR)}$, $\frac{4}{5}$

4. MN = 15 **5.** 20 cm **6.** $4\sqrt{2}$

7. $\boxed{PF}$; $\boxed{x}$ + $\boxed{2x}$; $\boxed{\angle FPQ}$; $\boxed{\angle FQP}$; $\frac{DF^2}{PF^2}$; $\boxed{20}$; $\boxed{45}$; $\boxed{45}$ – $\boxed{20}$; $\boxed{\text{25 sq. unit}}$

Problem set 1

1. (1) (B), (2) (B), (3) (B), (4) (D), (5) (A)

2. $\frac{7}{13}$, $\frac{7}{20}$, $\frac{13}{20}$ **3.** 9 cm **4.** $\frac{3}{4}$ **5.** 11 cm **6.** $\frac{25}{81}$ **7.** 4

8. PQ = 80, QR = $\frac{280}{3}$, RS = $\frac{320}{3}$ **9.** $\frac{\boxed{PM}}{\boxed{MQ}} = \frac{\boxed{PX}}{\boxed{XQ}}$, $\frac{\boxed{PM}}{\boxed{MR}} = \frac{\boxed{PY}}{\boxed{YR}}$,

10. $\frac{AX}{XY} = \frac{3}{2}$ **12.** $\frac{\boxed{3}}{\boxed{2}}$, $\frac{\boxed{3}+\boxed{2}}{\boxed{2}}$, $\frac{\boxed{5}}{\boxed{3}}$, $\boxed{AA}$, $\frac{\boxed{5}}{\boxed{3}}$, $\boxed{15}$

Chapter 2 Pythagoras Theorem

Practice set 2.1

1. Pythagorean triplets ; (1), (3), (4), (6) **2.** NQ = 6 **3.** QR = 20.5

4. RP = 12, PS = $6\sqrt{3}$

5. $\boxed{\text{side opoosite to congruent angles}}$, $\boxed{45°}$, $\boxed{\frac{1}{\sqrt{2}}}$, $\boxed{\frac{1}{\sqrt{2}}}$, $\boxed{\frac{1}{\sqrt{2}}}$, $\boxed{2}$

6. side = $5\sqrt{2}$ cm, perimeter = $20\sqrt{2}$ cm **7.** (1) 18 (2) $4\sqrt{13}$ (3) $6\sqrt{13}$ **8.** 37 cm

10. 8.2 metre.

Practice set 2.2

1. 12 **2.** $2\sqrt{10}$ **4.** 18 cm

Problem set 2

1. (1) (B), (2) (B), (3) (A), (4) (C), (5) (D), (6) (C), (7) (B), (8) (A).

2. (1) $a\sqrt{3}$, (2) form a right angled triangle. (3) 61 cm, (4) 15 cm, (5) $x\sqrt{2}$, (6) $\angle$ PRQ.

3. RS = 6 cm, ST = $6\sqrt{3}$ cm **4.** 20 cm **5.** side = 2 cm, perimeter = 6 cm

6. 7 **7.** AP = $2\sqrt{7}$ cm **10.** 7.5 km / hr **12.** 8 cm **14.** 8 cm

15. 192 sq.unit **17.** 58 **18.** 26

Chapter 3 Circle

Practice set 3.1

1. (1) 90°, tangent-radius theorem (2) 6 cm ; perpendicular distance
(3) $6\sqrt{2}$ cm (4) 45°

2. (1) $5\sqrt{3}$ cm (2) 30° (3) 60° **4.** 9 cm

Practice set 3.2

1. 1.3 cm 2. 9.7 cm **4.** (3) 110° **5.** $4\sqrt{6}$ cm

Practice set 3.3

1. m(arc DE) = 90°, m(arc DEF) = 160°

Practice set 3.4

1. (1) 60° (2) 30° (3) 60° (4) 300° **2.** (1) 70° (2) 220° (3) 110° (4) 55°

3. $\angle$R = 92°; $\angle$N = 88° **7.** 44° **8.** 121°

Practice set 3.5

1. PS = 18; RS = 10, **2.** (1) 7.5 (2) 12 or 6

3. (1) 18 (2) 10 (3) 5 **4.** 4

Problem set 3

1. (1) D (2) B (3) B (4) C (5) B (6) D (7) A (8) B (9) A (10) C.

2. (1) 9 cm (2) in the interior of the circle (3) 2 locations, 12 cm

3. (1) 6 (2) $\angle$K = 30°; $\angle$M = 60° **5.** 10 **6.** (1) 9 cm (2) 6.5 cm

(3) 90° ; MS : SR = 2 : 1 **9.** $4\sqrt{3}$ cm

13. (1) 180° (2) $\angle$ AQP $\cong \angle$ ASQ $\cong \angle$ ATQ

(3) $\angle$ QTS $\cong \angle$ SQR $\cong \angle$ SAQ (4) 65°, 130° (5) 100° **14.**(1) 70°

(2) 130° (3) 210° **15.** (1) 56° (2) 6 (3) 16 or 9 **16.** (1) 15.5°

(2) 3.36 (3) 6 **18.** (1) 68° (2) OR = 16.2, QR = 13 (3) 13 **21.** 13

Chapter 4 Geometric Constructions

Problem set 4

1. (1) C (2) A (3) A

Chapter 5 Co-ordinate Geometry

Practice set 5.1

1. (1) $2\sqrt{2}$ (2) $4\sqrt{2}$ (3) $\frac{11}{2}$ (4) 13 (5) 20 (6) $\frac{29}{2}$

2. (1) are collinear. (2) are not collinear. (3) are not collinear. (4) are collinear.

3. (−1, 0) **7.** 7 or −5

Practice set 5.2

1. (1, 3) **2.** (1) $\left(-\frac{1}{3},-\frac{1}{3}\right)$ (2) $\left(\frac{4}{7},-\frac{11}{7}\right)$ (3) $\left(0,\frac{13}{3}\right)$ **3.** 2:7 **4.** (−6, 3)

5. 2:5, k = 6 **6.** (11, 18) **7.** (1) (1, 3) (2) (6, −2) (3) $\left(\frac{19}{3},\frac{22}{3}\right)$

8. (−1, −7) **9.** h = 7, k = 18 **10.** (0, 2) ; (−2, −3)

11. (−9, −8), (−4, −6), (1, −4) **12.** (16, 12), (12, 14), (8, 16), (4, 18)

Practice set 5.3

1. (1) 1 (2) $\sqrt{3}$ (3) slope cannot be determined.

2. (1) 2 (2) $-\frac{3}{8}$ (3) $\frac{5}{2}$ (4) $\frac{5}{4}$ (5) $\frac{1}{2}$ (6) slope cannot be determined.

3. (1) are collinear. (2) are collinear. (3) are not collinear. (4) are collinear.

(5) are collinear. (6) are collinear.

4. -5; $\frac{1}{5}$; $-\frac{2}{3}$ 6. k = 5 7. k = 0 8. k = 5

Problem set 5

1. (1) D (2) D (3) C (4) C

2. (1) are collinear. (2) are collinear. (3) are not collinear. 3. (6, 13) 4. 3:1

5. (−7, 0)   6. (1) $a\sqrt{2}$ (2) 13   (3) 5a **7.** $\left(-\frac{1}{3}, \frac{2}{3}\right)$

8. (1) Yes, scalene triangle (2) No. (3) Yes, equilateral triangle   9. $k = 5$

13. 5, $2\sqrt{13}$, $\sqrt{37}$ **14.** (1, 3)   **16.** $\left(\frac{25}{6}, \frac{13}{6}\right)$, radius = $\frac{13\sqrt{2}}{6}$ **17.** (7, 3)

18. Parallelogram   **19.** A(20, 10), P(16, 12), R(8, 16), B(0, 20).   **20.** (3, −2)

21. (7, 6) and (3, 6)   **22.** 10 and 0

Chapter 6 Trigonometry

Practice set 6.1

1. $\cos\theta = \frac{24}{25}$; $\tan\theta = \frac{7}{24}$ **2.** $\sec\theta = \frac{5}{4}$; $\cos\theta = \frac{4}{5}$

3. $\operatorname{cosec}\theta = \frac{41}{9}$; $\sin\theta = \frac{9}{41}$ **4.** $\sec\theta = \frac{13}{5}$; $\cos\theta = \frac{5}{13}$; $\sin\theta = \frac{12}{13}$

5. $\frac{\sin\theta + \cos\theta}{\sec\theta + \operatorname{cosec}\theta} = \frac{1}{2}$

Practice set 6.2

1. Height of the church is 80 metre.
2. The ship is 51.90 metre away from the ligthhouse.
3. Height of the second building is $(10 + 12\sqrt{3})$ metre.
4. Angle made by the wire with the horizontal line is 30°.
5. Height of the tree is $(40 + 20\sqrt{3})$ metre.
6. The length of the string is 69.20 metre.

Problem set 6

1. (1) A   (2) B   (3) C   (4) A

2. $\cos 60 = \frac{60}{61}$ **3.** $\sin\theta = \frac{2}{\sqrt{5}}$; $\cos\theta = \frac{1}{\sqrt{5}}$; $\operatorname{cosec}\theta = \frac{\sqrt{5}}{2}$; $\sec\theta = \sqrt{5}$; $\cot\theta = \frac{1}{2}$

4. $\sin\theta = \frac{5}{13}$; $\cos\theta = \frac{12}{13}$; $\operatorname{cosec}\theta = \frac{13}{5}$; $\tan\theta = \frac{5}{12}$; $\cot\theta = \frac{12}{5}$

6. Height of the building is $16\sqrt{3}$ metre.
7. The ship is $100\sqrt{3}$ metre away from the ligthhouse.
8. Height of the second buliding is $(12 + 15\sqrt{3})$ metre.
9. The maximum height that ladder can reach is 20.80 metre.

10. the plane was 1026 metre high at the time of landing.

Chapter 7 Mensuration

Practice set 7.1

1. 11.79 cm^3 **2.** 113.04 cm^3 **3.** 1413 sq.cm (by taking π = 3.14) **4.** 616 sq.cm

5. 21 cm **6.** 12 jugs **7.** 5 cm **8.** 273π sq.cm **9.** 20 tablets

10. 94.20 cm^3, 103.62 sq.cm **11.** 5538.96 sq.cm, 38772.72 cm^3

12. 1468.67π cm^3

Practice set 7.2

1. 10.780 litre **2.** (1) 628 sq.cm (2) 1356.48 sq.cm (3) 1984.48 cm^3

Practice set 7.3

1. 47.1 sq.cm **2.** 25.12 cm **3.** 3.85 sq.cm **4.** 214 sq.cm **5.** 4 cm

6. (1) 154 sq.cm (2) 25.7 sq.cm (3) 128.3 sq.cm **7.** 10.2 sq.cm

8. 7.3 cm ; 22 cm **9.** (1) $90°$ (2) 22 cm

10.(1) 12.83 sq.cm (2) 89.83 sq.cm (3) 115.5 sq.cm **11.** 3.5 cm

12. x = 154 sq.cm ; y = 38.5 sq.cm ; z = 101.5 sq.cm

13. (1) 84.87 sq.cm (2) 25.67 sq.cm (3) 77.01 sq.cm (4) 7.86 sq.cm

Practice set 7.4

1. 3.72 sq.cm **2.** 9.08 sq.cm **3.** 0.65625 sq.unit **4.** 20 cm

5. 20.43 sq.cm ; 686.07 sq.cm

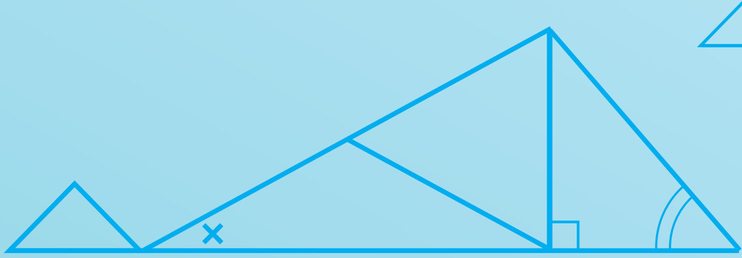
Problem set 7

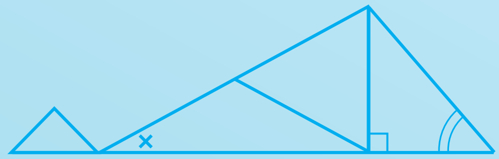
1. (1) A, (2) D, (3) B, (4) B, (5) A, (6) A, (7) D, (8) C.

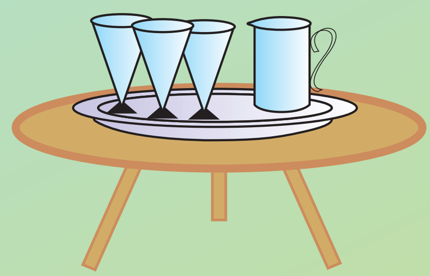
2. 20.35 litre **3.** 7830 balls **4.** 2800 coins (by taking $\pi = \frac{22}{7}$) **5.** Rs. 6336

6. 452.16 sq.cm ; 3385.94 gm **7.** 2640 sq.cm **8.** 108 metre

9. $150°$; 5π cm **10.** 39.28 sq.cm







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