Solved Question Paper
2017
Class – XII
Subject – Mathematics
All India

Time allowed: 3 hours **Maximum Marks: 100**

General Instructions:

(i) All questions are compulsory.

(*ii*) The question paper consists of 29 questions divided into four sections A, B,

C and D. Section A comprises of 4 questions of one mark each, Section B comprises of 8 questions of two marks each, Section C comprises of 11 questions of four marks each and Section D comprises of 6 questions of six marks each.

(*iii*) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(*iv*) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(*v*) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

SECTION A

Question 1:

If for any 2×2 square matrix A, A (adj A) $=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$, then write the value of $|A|$.

Solution:

Given,

$$A(\text{adj } A)=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}\Rightarrow A(\text{adj } A)=8\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\Rightarrow A(\text{adj } A)=8I_2 \qquad ...(1)$$

Now, for any square matrix A of order 2, we have

$$A(\text{adj } A)=|A|I_2 \qquad ...(2)$$

Comparing …(1) and …(2), we have

$|A| = 8.$

Question 2:

Determine the value of '*k*' for which the following function is continuous at *x* = 3:

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Solution:

Given that the function *f*(*x*) is continuous at *x* = 3 iff

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

$$\Rightarrow f(3) = \lim_{x \to 3^+} f(x)$$

$$\Rightarrow \lim_{x \to 3^+} \frac{(x+3)^2 - 36}{x-3}$$

$$\Rightarrow k = \lim_{x \to 3^+} \frac{(x+3-6)(x+3+6)}{x-3}$$

$$\Rightarrow k = \lim_{x \to 3^+} \frac{(x-3)(x+9)}{x-3}$$

$$\Rightarrow k = \lim_{x \to 3^+} (x+9)$$

$$\Rightarrow k = 3 + 9 = 12$$

$$\Rightarrow k = 12.$$

Question 3:

Find: $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

Solution:

Let, $I = \int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

$\Rightarrow I = \int \frac{\sin^2 x}{\sin x \cos x} dx - \int \frac{\cos^2 x}{\sin x \cos x} dx$

$\Rightarrow I = \int \tan x dx - \int \cot x dx$

$\Rightarrow I = \log|\sec x| - \log|\sin x| + C$

Question 4:

Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

Solution:

$5x - 2.5y + 5z = 20$ can also be written as $2x - y + 2z = 8$.

If we carefully observe the equations of both the planes, i.e., $2x - y + 2z = 5$ and $2x - y + 2z = 8$.

So the above two planes are parallel planes.

For two parallel planes, $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$

Distance between them is given by:

$$d = \left| \frac{(d_2 - d_1)}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Given planes are $2x - y + 2z - 5 = 0$ and $2x - y + 2z - 8 = 0$

Therefore the distance between them is:

$$d = \left| \frac{(-5+8)}{\sqrt{2^2 + (-1)^2 + 2^2}} \right| = \left| \frac{3}{\sqrt{4+1+4}} \right| = \left| \frac{3}{3} \right| = 1 \text{ unit.}$$

SECTION B

Question 5:

If A is skew-symmetric matrix of order 3, then prove that det A = 0.

Solution:

Given, A is a skew-symmetric matrix of order 3.

Therefore, $A^T = -A$

$\det(A^T) = \det(A)$

We know that, $\det(kA) = k^n \det|A|$, where n is the order of the matrix

$\Rightarrow \det A = (-1)^3 \det A$
$\Rightarrow \det A = -\det A$
$\Rightarrow \det A + \det A = 0$
$\Rightarrow 2\det A = 0$
$\Rightarrow \det A = 0.$

Question 6:

Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $\left[-\sqrt{3}, 0\right]$.

Solution:

Let f be a real valued function defined on the closed interval [a, b] such that it is continuous on [a, b] and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

The given function is $f(x) = x^3 - 3x$.

It is a polynomial function so, it is continuous and differentiable everywhere, therefore $f(x)$ is continuous on $\left[-\sqrt{3}, 0\right]$ and differentiable on $\left(-\sqrt{3}, 0\right)$.

Also, $f\left(-\sqrt{3}\right) = \left(-\sqrt{3}\right)^3 - 3\left(-\sqrt{3}\right) = -3\sqrt{3} + 3\sqrt{3} = 0$

and $f(0) = (0)^3 - 3 \times 0 = 0$

Since all the three conditions of Rolle's Theorem are satisfied. So there exists a point $c \in \left(-\sqrt{3}, 0\right)$ such that $f'(c) = 0$

$$\text{Given, } f(x) = x^2 - 3x$$
$$\Rightarrow f'(x) = 3x^2 - 3$$
$$\therefore f'(c) = 0$$
$$\Rightarrow 3c^2 - 3 = 0$$
$$\Rightarrow c^2 - 1 = 0$$
$$\Rightarrow (c+1)(c-1) = 0$$
$$\Rightarrow c = -1 \text{ or } c = 1$$

$$\text{Now, } c \neq 1 \qquad \left[\text{because, } 1 \notin \left(-\sqrt{3}, 0\right)\right]$$

$$\therefore c = -1, \text{ where } c \in \left(-\sqrt{3}, 0\right)$$

Therefore, the required value of c is – 1.

Question 7:

The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm?

Solution:

Suppose, x be the length of one side, V be the volume and S be the surface area of the cube.

$$\text{Now, } V = x^3 \quad \ldots(1)$$

$$S = 6x^2 \text{ where } x \text{ is a function of time } t \quad \ldots(2)$$

Given that,

$$\frac{dV}{dt} = 9 \text{ cm}^3/\text{sec}$$

$$\Rightarrow 9 = \frac{dV}{dt} = \frac{d\left(x^3\right)}{dt} = 3x^2 . \frac{dx}{dt} \qquad \text{(using chain rule)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{3}{x^2} \qquad \ldots(3)$$

Now,

$$S = 6x^2$$

$$\Rightarrow \frac{dS}{dt} = 6\,(2x)\,\frac{dx}{dt}$$
$$\Rightarrow \frac{dS}{dt} = 6\,(2x)\,\frac{3}{x^2}$$
$$\Rightarrow \frac{dS}{dt} = 6\,(2)\,\frac{3}{x}$$
$$\Rightarrow \frac{dS}{dt} = 6\,(2)\,\frac{3}{10}$$

Hence, for $x = 10$ cm , $\frac{dS}{dt} = 3.6 \text{ cm}^2/\text{sec}$.

Question 8:

Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

Solution:

$$f(x) = x^3 - 3x^2 + 6x - 100$$
$$\Rightarrow f'(x) = 3x^2 - 6x + 6$$
$$\Rightarrow f'(x) = 3(x^2 - 2x + 2)$$
$$\Rightarrow f'(x) = 3(x^2 - 2x + 1) + 3$$
$$\Rightarrow f'(x) = 3(x-1)^2 + 3$$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

Now, $3(x-1)^2 \geq 0 \forall x \in R$

$$\Rightarrow 3(x-1)^2 + 3 > 0 \forall x \in R$$
$$\Rightarrow f'(x) > 0 \forall x \in R$$

Hence, the given function is increasing on *R*.

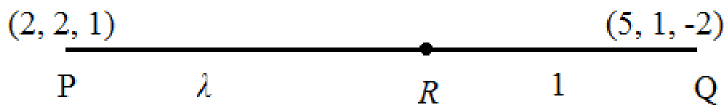
Question 9:

The *x* – coordinate of a point on the line joining the points P (2, 2, 1) and Q (5, 1, –2) is 4. Find its *z* – coordinate

Solution:

Suppose the point R divide PQ in the ratio $\lambda:1$. Then, the coordinates of R will be

$$\left(\frac{5\lambda+2}{\lambda+1},\frac{\lambda+2}{\lambda+1},\frac{-2\lambda+1}{\lambda+1}\right)$$



It is given that the *x*-coordinate of R is 4.

Therefore,

$$\frac{5\lambda+2}{\lambda+1}=4$$
$$\Rightarrow 5\lambda+2=4\lambda+4$$
$$\Rightarrow 5\lambda-4\lambda=4-2$$
$$\Rightarrow \lambda=2$$

As, z - coordinate of R is $\frac{-2\lambda+1}{\lambda+1}$

So, putting, $\lambda=2$ in $\frac{-2\lambda+1}{\lambda+1}$,

z-coordinate of R = $\frac{-2\times2+1}{2+1}=\frac{-3}{3}=-1$.

Question 10:

A die, whose faces are marked 1, 2, 3, in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

Solution:

Total number of outcomes = 6

Let event

A: Number obtained is even. The outcomes in favour of the event A are 2, 4, 6.

$$\therefore P(A)=\frac{3}{6}=\frac{1}{2}$$

B: Number obtained is red. The outcomes favour of the event B are 1, 2, 3.

$$\therefore P(B)=\frac{3}{6}=\frac{1}{2}$$

Now,

$$P(A)P(B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$

Also, $P(A\cap B)=P(\text{getting an even red number})=\frac{1}{6}$

As, $P(A\cap B)\neq P(A)P(B)$, hence, the events A and B are not independent events.

Question 11:

Two tailors, A and B earn Rs 300 and Rs 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

Solution:

Let A work for x days and tailor B work for y days.

In one day. A can stitch 6 shirts and 4 pairs of trousers whereas B can stitch 10 shirts and 4 pairs of trousers.

In x days A can stitch $6x$ shirts and $4x$ pairs of trousers, similarly, in y days B can stitch 10 y shirts and 4 y pairs of trousers.

It is given that the minimum requirement of the shirts and pairs of trousers are respectively 60 and 32 respectively.

Thus,

$$6x+10y\geq 60$$
$$4x+4y\geq 32$$

According to the question, A and B earn Rs 300 and Rs 400 per day respectively. Thus, in x days and y days A and B earn Rs $300x$ and Rs $400y$ respectively.

Let Z denotes the total cost.

$$\therefore Z=Rs(300x+400y)$$

Number of days cannot be negative.

Therefore,

$x \geq 0,\ y \geq 0$

Hence, the required LPP is as follows:

Minimize Z = 300x + 400y

Subject to

$$6x + 10y \geq 60$$
$$4x + 4y \geq 32$$
$$x \geq 0,\ y \geq 0$$

Question 12:

Find:

$$\int \frac{dx}{5 - 8x - x^2}$$

Solution:

$$\int \frac{dx}{5 - 8x - x^2}$$

$$= \int \frac{dx}{5 - \left(8x - x^2\right)}$$

$$= \int \frac{dx}{5 - \left(x^2 + 8x + 16 - 16\right)}$$

$$= \int \frac{dx}{21 - (x+4)^2}$$

$$= \int \frac{dx}{\left(\sqrt{21}\right)^2 - (x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21} + (x+4)}{\sqrt{21} - (x+4)}\right| + c$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4}\right| + c$$

SECTION C

Question 13:

If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$, then find the value of x.

Solution:

$$\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4}=\frac{\pi}{4}-\tan^{-1}\frac{x+3}{x+4}$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4}=\tan^{-1}1-\tan^{-1}\frac{x+3}{x+4}$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4}=\tan^{-1}\left(\frac{1-\frac{x+3}{x+4}}{1+\frac{x+3}{x+4}}\right) \qquad \left[\because \tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4}=\tan^{-1}\left(\frac{\frac{x+4-x-3}{x+4}}{1+\frac{x+3}{x+4}}\right)$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4}=\tan^{-1}\left(\frac{1}{2x+7}\right)$$

$$\Rightarrow \frac{x-3}{x-4}=\frac{1}{2x+7}$$

$$\Rightarrow (x-3)(2x+7)=x-4$$

$$\Rightarrow 2x^2+x-21=x-4$$

$$\Rightarrow 2x^2=17$$

$$\Rightarrow x=\pm\sqrt{\frac{17}{2}}$$

Question 14:

Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}=(a-1)^3$$

OR

Find matrix A such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

Solution:

Suppose,

$$\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Applying, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have

$$\Delta = \begin{vmatrix} a^2+2a & 2a-2 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} (a+3)(a-1) & 2(a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = (a-1)^2 \begin{vmatrix} (a+3) & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding alogn C_3

$$\Rightarrow \Delta = (a-1)^2\left[0-0+1\times(a+3-4)\right]$$

$$\Rightarrow \Delta = (a-1)^3$$

OR

$$\text{Let} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Since the product of two matrices is a 3 × 2 matrix and pre multiplier of A is a 3 ×2 matrix so, A is a 2 × 2 matrix.

Let $A=\begin{bmatrix} x\, y \\ a\, b \end{bmatrix}$

Therefore,

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix}=\begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

On equating the element of LHS with corresponding elements of RHS, we get

$$\begin{aligned} &2a-c=-1 && ...(1) \\ &2b-d=-8 && ...(2) \\ &a=1 && ...(3) \\ &b=-2 && ...(4) \end{aligned}$$

From …(1), …(2), …(3) and (4)

$$A=\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

Question 15:

If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$.

OR

If $e^y(x+1)=1$, then show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Solution:

Suppose, $u=x^y$ and $v=y^x$

$\Rightarrow u+v=a^b$

Differentiating both sides with respect to *x*, we have

$$\frac{du}{dx}+\frac{dv}{dx}=0.....(1)$$

Now,

$$u=x^y$$

Taking logarithm on both the sides

$$\Rightarrow \log u = y\log x$$

Differentiating both sides with respect to *x*, we have

$$\frac{1}{u}\frac{du}{dx}=y\times\frac{1}{x}+\log x\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx}=u\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{du}{dx}=x^y\left(\frac{y+x\log x\dfrac{dy}{dx}}{x}\right)$$

$$\Rightarrow \frac{du}{dx}=x^{y-1}\left(y+x\log x\frac{dy}{dx}\right) \qquad .....(2)$$

Now,

$$v=y^x$$

Taking logarithm on both the sides

$$\Rightarrow \log v = x\log y$$

Differentiating both sides with respect to *x*, we have

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=x\times\frac{1}{y}\times\frac{dy}{dx}+\log y$$

$$\Rightarrow \frac{dy}{dx}=v\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right)$$

$$\Rightarrow \frac{dy}{dx}=y^x\left(\frac{x\times\dfrac{dy}{dx}+y\log y}{y}\right)$$

$$\Rightarrow \frac{dy}{dx}=y^{x-1}\left(x\frac{dy}{dx}+y\log y\right) \qquad ....(3)$$

Form …(1), …(2) and …(3), we have

$$x^{y-1}\left(y+x\log x\frac{dy}{dx}\right)+y^{x-1}\left(x\frac{dy}{dx}+y\log y\right)=0$$

$$\Rightarrow\left(x^{y}\log x-xy^{x-1}\right)\frac{dy}{dx}=y^{x}\log y-x^{y-1}y$$

$$\Rightarrow\frac{dy}{dx}=\frac{y^{x}\log y-x^{y-1}y}{x^{y}\log x-xy^{x-1}}$$

OR

The given equation is $e^{y}\left(x+1\right)=1$.

$$\Rightarrow e^{y}.1+(x+1).e^{y}.\frac{dy}{dx}=0$$

$$\Rightarrow\frac{dy}{dx}=\frac{-1}{\left(x+1\right)}$$

$$\Rightarrow\frac{d^{2}y}{dx^{2}}=\frac{-1}{\left(x+1\right)^{2}}=\left(\frac{dy}{dx}\right)^{2}$$

Question 16:

Find:

$$\int\frac{\cos\theta}{\left(4+\sin^{2}\theta\right)\left(5-4\cos^{2}\theta\right)}d\theta$$

Solution:

$$\text{Let, } I=\int\frac{\cos\theta}{\left(4+\sin^{2}\theta\right)\left(5-4\cos^{2}\theta\right)}d\theta$$

$$=\int\frac{\cos\theta}{\left(4+\sin^{2}\theta\right)\left[5-4\left(1-\sin^{2}\theta\right)\right]}d\theta$$

$$=\int\frac{\cos\theta}{\left(4+\sin^{2}\theta\right)\left(1+4\sin^{2}\theta\right)}d\theta \qquad \text{...(1)}$$

Let

$$\sin\theta=t$$

$$\Rightarrow\cos\theta d\theta=dt$$

So, $I=\int \frac{dt}{\left(4+t^2\right)\left(1+4t^2\right)}$ ....(2)

Suppose, $t^2=y$

$$I=\int \frac{dt}{\left(4+t^2\right)\left(1+4t^2\right)}=\frac{1}{(4+y)(1+4y)}$$

Applying partial fraction we get,

$$\frac{1}{(4+y)(1+4y)}=\frac{A}{(4+y)}+\frac{B}{(1+4y)}$$

$$\Rightarrow 1=(1+4y)+B(4+y)$$
$$\Rightarrow A=\frac{-1}{15},B\frac{4}{15}$$

Putting these values in … (2) we get,

$$I=\int\left[\frac{\frac{-1}{15}}{4+t^2}+\frac{\frac{4}{15}}{1+4t^2}\right]dt$$

$$\Rightarrow I=\int\frac{-1}{15}\left[\frac{1}{4+t^2}\right]dt+\int\frac{4}{15}\left[\frac{1}{1+4t^2}\right]dt$$
$$\Rightarrow I=\frac{-1}{15}\int\frac{dt}{4+t^2}+\frac{4}{15}\int\frac{dt}{1+4t^2}$$
$$\Rightarrow I=\frac{-1}{15}\int\frac{dt}{4+t^2}+\frac{4}{15}\int\frac{dt}{4\left(\frac{1}{4}+t^2\right)}$$
$$\Rightarrow I=\frac{-1}{15}\int\frac{dt}{4+t^2}+\frac{1}{15}\int\frac{dt}{\left(\frac{1}{4}+t^2\right)}$$
$$\Rightarrow I=\frac{-1}{15}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]+\frac{1}{15}\left[2\tan^{-1}2t\right]+c$$

Putting, $t=\sin\theta$, we have

$$\Rightarrow I=\frac{-1}{15}\left[\frac{1}{2}\tan^{-1}\left(\frac{\sin\theta}{2}\right)\right]+\frac{1}{15}\left[2\tan^{-1}2\sin\theta\right]+c$$
$$\Rightarrow I=\frac{-1}{30}\tan^{-1}\left[\frac{\sin\theta}{2}\right]+\frac{2}{15}\tan^{-1}\left(2\sin\theta\right)+c$$

Question 17:

Evaluate:

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

OR

Evaluate:

$$\int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$$

Solution:

$$\text{Let, } I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} \qquad \ldots(1)$$

$$\text{As, } \int_0^{\pi} f(x)\,dx = \int_0^{a} (a-x)\,dx$$

$$\therefore I = \int_0^{\pi} \left\{ \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{(\pi - x)\tan x}{(\sec x + \tan x)} \right\} dx \qquad \ldots(2)$$

Adding equations … (1) and … (2), we have

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1.dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi\int_0^{\pi}\left(\sec^2 x - \tan x \sec x\right)dx$$

$$\Rightarrow 2I = \pi^2 - \pi\left[\tan x - \sec x\right]_0^{x}$$

$$\Rightarrow 2I = \pi^2 - \pi\left[\tan \pi - \sec \pi - \tan 0 + \sec 0\right]$$

$$\Rightarrow 2I = \pi^2 - \pi\left[0 - (-1) - 0 + 1\right]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

OR

Suppose $I = \int_1^4 \left(|x-1| + |x-2| + |x-4|\right)dx$

$$\Rightarrow I = \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-4|\,dx$$

As,

$$|x-1| = \begin{cases} -(x-1), & x \le 1 \\ x-1, & 1 \le x \le 4 \end{cases}$$

$$|x-2| = \begin{cases} (x-2)-, & 1 \le x \le 2 \\ x-2, & 2 \le x \le 4 \end{cases}$$

$$|x-4| = \begin{cases} -(x-4)-, & 1 \le x \le 4 \\ x-4, & x > 4 \end{cases}$$

$$\therefore I = \int_1^4 (x-1)\,dx - \int_1^2 (x-2)\,dx + \int_2^4 (x-2)\,dx - \int_1^4 (x-4)\,dx$$

$$\Rightarrow I = \left[\frac{x^2}{2} - x\right]_1^4 - \left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 - \left[\frac{x^2}{2} - 4x\right]_1^4$$

$$\Rightarrow I = \left[(8-4) - \left(\frac{1}{2} - 1\right)\right] - \left[(2-4) - \left(\frac{1}{2} - 2\right)\right] + \left[(8-8) - (2-4)\right] - \left[\frac{1}{2} - 4\right]$$

$$\Rightarrow I = \frac{23}{2}$$

Question 18:

Solve the differential equation $\left(\tan^{-1} x - y\right)dx = \left(1+x^2\right)dy$.

Solution:

$$\left(\tan^{-1} x - y\right)dx = \left(1+x^2\right)dy$$

Rearranging, we have

$$\left(1+x^2\right)\frac{dy}{dx} + y = \tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{1} x}{1+x^2}$$

It is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Where,

$$P = \frac{1}{1+x^2}$$

$$Q = \frac{\tan^{-1} x}{1+x^2}$$

$$\therefore I.F. = e^{\int Pdx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1} x}$$

The solution of the linear differential equation is given by

$$y \times I.F = \int Q \times I.Fdx$$

$$ye^{\tan^{-1} x} = \int e^{\tan^{-1} x} \frac{\tan^{-1} x}{1+x^2} dx \quad ...(i)$$

Let,

$$I = \int e^{\tan^{-1} x} \frac{\tan^{-1} x}{1+x^2} dx$$

Putting, $\tan^{-1} x = t$, we get

$$\Rightarrow \frac{1}{1+x^2}dx = dt$$
$$\therefore I = \int te^t dt$$
$$\Rightarrow I = t\int e^t dt - \int\left[\frac{d}{dt}(t)\int e^t dt\right]dt$$
$$\Rightarrow I = te^t - e^t + C$$
$$\Rightarrow I = (t-1)e^t + C$$
$$\Rightarrow I = \left(\tan^{-1}x - 1\right)e^{\tan^{-1}x} + C$$

Putting the value of I in …(i), we have

$$e^{\tan^{-1}x}y = \left(\tan^{-1}x - 1\right)e^{\tan^{-1}x} + C$$
$$\Rightarrow y = \tan^{-1}x - 1 + C\,e^{-\tan^{-1}x}$$

Hence, $y = \tan^{-1}x - 1 + C\,e^{-\tan^{-1}x}$ is the required solution of the given differential equation.

Question 19:

Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{j}$ respectively are the vertices of a right-angled triangle. Hence find the area of the triangle.

Solution:

Given,

$2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{j}$ are the position vectors of points A, B and C respectively.

$$\therefore \overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

AB, BC and CA are not parallel, also $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

Therefore, A, B and C are vertices of a triangle.

Also, $\overrightarrow{BC}.\overrightarrow{CA} = 0$, *so*, A, B and C form a right triangle

Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{210}$ square units..

Question 20:

Find the value of λ, if four points with position vectors $3\hat{i}+6\hat{j}+9\hat{k}, \hat{i}+2\hat{j}+3\hat{k}, 2\hat{i}+3\hat{j}+\hat{k}$ and $4\hat{i}+6\hat{j}+\lambda\hat{k}$ are coplanar.

Solution:

Suppose, A $(3\hat{i}+6\hat{j}+9\hat{k})$, B $(\hat{i}+2\hat{j}+3\hat{k})$, C $(2\hat{i}+3\hat{j}+\hat{k})$ and D $(4\hat{i}+6\hat{j}+\lambda\hat{k})$ be the given point.

The given points will be coplanar if vectors are $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ coplanar

$$\overrightarrow{AB} = (\hat{i}+2\hat{j}+3\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k}) = -2\hat{i}-4\hat{j}-6\hat{k}$$
$$\overrightarrow{AC} = (2\hat{i}+3\hat{j}+\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k}) = -\hat{i}-3\hat{j}-8\hat{k}$$

$$\overrightarrow{AD} = (4\hat{i}+6\hat{j}+\lambda\hat{k})-(3\hat{i}+6\hat{j}+9\hat{k}) = \hat{i}+(\lambda-9)\hat{k}$$

So,

$$\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = 0$$

$$\Rightarrow \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda-9 \end{vmatrix} = 0$$

$$\Rightarrow -2\left(-3\lambda+27\right)+4\left(-\lambda+9+8\right)-6\left(3\right)=0 \Rightarrow \lambda = 2$$

Question 21:

There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

Solution:

Let X denotes the sum of the number on the drawn cards.

According to the question, X can take the values 4, 6, 8, 10 and 12.

P $(X = 4)$ = Probability of getting 4 as sum

P $(X = 4)$ = P [(Getting 1 in the first draw and 3 in the second draw) or (Getting 3 in the first draw and 1 in the second draw)]

$$=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{2}{12}=\frac{1}{6}$$

Similarly,

$$P(X=6)=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{2}{12}=\frac{1}{6}$$

$$P(X=8)=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{4}{12}=\frac{2}{6}$$

$$P(X=10)=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{2}{12}=\frac{1}{6}$$

$$P(X=12)=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{2}{12}=\frac{1}{6}$$

Thus, the probability distribution of X is given below:

X	4	6	8	10	12
P (X)	1/6	1/6	2/6	1/6	1/6

Now,

$$\sum xP(X)=\frac{1}{6}\times 4+\frac{1}{6}\times 6+\frac{2}{6}\times 8+\frac{1}{6}\times 10+\frac{1}{6}\times 12=\frac{1}{6}\times(4+6+16+10+12)=\frac{1}{6}\times 48=8$$

$$\sum x^2P(X)=\frac{1}{6}\times 16+\frac{1}{6}\times 36+\frac{2}{6}\times 64+\frac{1}{6}\times 100+\frac{1}{6}\times 144=\frac{424}{6}$$

$$\therefore \text{ Mean } =\sum xP(X)=8$$

$$\text{Variance}=\sum x^2P(X)-\sum[xP(X)]^2=\frac{424}{6}-64=\frac{20}{3}.$$

Question 22:

Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

Solution:

Suppose,

E_1: Selecting a student with 100% attendance

E_2: Selecting a student who is not regular

A: selected student attains grade A

Therefore,

$$P(E_1)=\frac{30}{100} \text{ and } P(E_2)=\frac{70}{100}$$

$$P\left(\frac{A}{E_1}\right)=\frac{70}{100} \text{ and } P\left(\frac{A}{E_2}\right)=\frac{10}{100}$$

$$P\left({E_1}/{A}\right)=\frac{P(E_1).P\left({A}/{E_1}\right)}{P(E_1).P\left({A}/{E_1}\right)+P(E_2).P\left({A}/{E_2}\right)}=\frac{\frac{30}{100}\times\frac{70}{100}}{\frac{30}{100}\times\frac{70}{100}+\frac{70}{100}\times\frac{10}{100}}=\frac{3}{4}$$

Using Baye's theorem, we have

Required probability that the student has 100% attendance given that he was found to have an A grade $=P\left(\frac{E_1}{A}\right)$

$$=\frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$=\frac{\frac{30}{100}\times\frac{70}{100}}{\frac{30}{100}\times\frac{70}{100}+\frac{70}{100}\times\frac{10}{100}}=\frac{3}{4}$$

Regularity is essential for progress in every domain of life i.e., school, college and office.

Question 23:

Maximise, Z = $x + 2y$

Subject to the constraints

$x + 2y \geq 100$
$2x - y \leq 0$
$2x + y \leq 200$
$x, y \geq 0$

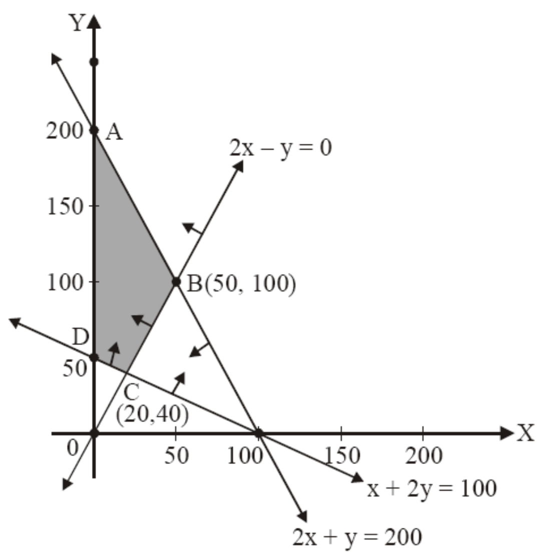
Solve the above LPP graphically.

Solution:

Given constraints are,

$x + 2y \geq 100,$
$2x + y \leq 200,$
$2x - y \leq 0,$
$x \geq 0, y \geq 0.$

Converting these inequations into equations and solving, the feasible region determined by the system of constraints is shown below:



The corner points of the feasible region are A (0, 200), B (50, 100), C (20, 40) and D (0, 50)

The values of Z at these corner points are as follows:

Corner Point	Value of objective function $Z = x + 2y$
A (0, 50)	Z = 0 + 400 = 400 (Maximum)
B (50, 100)	Z = 50 + 200 = 250
C (20, 40)	Z = 20 + 80 = 100
D (0, 50)	Z = 0 + 100 = 100

Maximum value of the objective function Z is 400 which is obtained at $x = 0$ and $y = 200$.

SECTION D

Question 24:

Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x-y+z+4, x-2y-2z=9, 2x+y+3z=1.$

Solution:

Let,

$$A=\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \text{and } C=\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

Now,

$$CA=\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow CA=\begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$\Rightarrow CA=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow CA=8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow CA = 8I_3$$

$$\Rightarrow \frac{1}{8}CA = I_3$$

$$\Rightarrow \left(\frac{1}{8}C\right)A = I_3$$

$$\Rightarrow A^{-1} = \frac{1}{8}C$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

or AX = B, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

The solution of this system of equations is given by

$$X = A^{-1}B$$

$$\Rightarrow X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 3, y = -2 \text{ and } z = -1$$

Question 25:

Consider *f*: $\mathbb{R}-\left\{-\frac{4}{3}\right\}\to R-\left\{\frac{4}{3}\right\}$ given by $f\left(x\right)=\frac{4x+3}{3x+4}$. Show that *f* is objective. Find the inverse of f and hence find $f^{-1}\left(0\right)$ and *x* such that $f^{-1}\left(x\right)=2$.

OR

Let $A=\mathbb{Q}\times\mathbb{Q}$ and let *be a binary operation on A defined by (*a*, *b*) * (*c*, *d*) = (*ac*, *b* + *a d*) for (*a*, *b*), (*c*, *d*) $\in A$. Determine, whether * is commutative and associative. Then, with respect to * on A

(*i*) Find the identity in A.

(*ii*) Find the invertible elements of A.

Solution:

Give, function $f:R-\left\{-\frac{4}{3}\right\}\to R-\left\{\frac{4}{3}\right\}$ is given by *f*(*x*) $=\frac{4x+3}{3x+4}$

For injectivity:

Suppose $x, y\in R-\left\{-\frac{4}{3}\right\}$ be such that

$$f\left(x\right)=f\left(y\right)$$

$$\Rightarrow\frac{4x+3}{3x+4}=\frac{4y+3}{3y+4}$$

$$\Rightarrow 12xy+9y+16x+12=12xy+9x+16y+12$$

$$\Rightarrow 7x=7y$$

$$\Rightarrow x=y$$

Hence, *f* is one – one function.
For Surjectivity:

Let y be an arbitrary element of $R-\left\{\frac{4}{3}\right\}$ then,

$$f(x) = y$$
$$\Rightarrow y = \frac{4x+3}{3x+4}$$
$$\Rightarrow 3xy + 4y = 4x + 3$$
$$\Rightarrow 4x - 3xy = 4y - 3$$
$$\Rightarrow x = \frac{4y-3}{4-3y}$$

As, $\forall y \in R - \left\{\frac{4}{3}\right\}, x \in R$

Therefore, f is ONTO and so it is bijective.

and $f^{-1}(y) = \frac{4y-3}{4-3y};\ y \in R - \left\{\frac{4}{3}\right\}$

$$f^{-1}(0) = -\frac{3}{4}$$

and $f^{-1}(x) = 2$

$$\Rightarrow \frac{4x-3}{4-3x} = 2$$
$$\Rightarrow 4x - 3 = 8 - 6x$$
$$\Rightarrow x = \frac{11}{10}$$

OR

Suppose $A = \mathbb{Q} \times \mathbb{Q}$ and * be a binary operation on A defined by

$(a,b)*(c,d) = (ac, b+ad); (a,b),(c,d) \in A.$

$(c,d)*(a,b) = (ca, d+bc)$

As, $b + ad \neq d + bc$

$\Rightarrow *$ is NOT commutative

For Associativity, we have

$[(a,b)*(c,d)]*(e,f) = (ac, b+ad)*(e,f) = (ace, b+ad+acf)$

$(a,b)*[(c,d)*(e,f)] = (a,b)*(ce, d+cf) = (ace, b+ad+acf)$

$\Rightarrow *$ is associative.

(i) Let (e, f) be the identity element in A

Then $(a, b) * (e, f) = (a, b) = (e, f) * (a, b)$

$\Rightarrow (ae, b + af) = (a, b) = (ae, f + be)$

⇒ $e = 1$, $f = 0$
⇒ (1, 0) is the identity element

(ii) Let (c, d) be the inverse element for (a, b)
⇒ $(a, b) * (c, d) = (1, 0) = (c, d) * (a, b)$
⇒ $(ac, b + ad) = (1, 0) = (ac, d + bc)$
⇒ $ac = 1$
⇒ $c = 1/a$ and $b + ad = 0$
⇒ $d = -b/a$ and $d + bc = 0$
⇒ $d = -bc$
⇒ $d = -b\,(1/a)$

⇒ $(1/a, -b/a)$, $a \neq 0$ is the inverse of $(a, b) \in$ A.

Question 26:

Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Solution:

Suppose, a closed cuboid with length x, breadth x and height y have fixed volume V.

Let S be the surface area of the cuboid.

Then,

$$V = x^2 y \; ....(1) \text{ and, } S = 2\left(x^2 + xy + xy\right) = 2x^2 + 4xy \qquad ...(2)$$

$$\text{Now, } S = 2x^2 + 4xy$$

$$\Rightarrow S = 2x^2 + 4x\frac{V}{x^2}$$

$$\Rightarrow S = 2x^2 + 4\frac{V}{x}$$

$$\Rightarrow \frac{ds}{dx} = 4x - 4\frac{V}{x^2} \qquad ....(3)$$

For maximum or minimum value, we have

$$\Rightarrow \frac{ds}{dx}=0$$
$$\Rightarrow 4x-4\frac{V}{x^2}=0$$
$$\Rightarrow V=x^3$$
$$\Rightarrow x^2y=x^3$$
$$\Rightarrow x=y$$

Differentiating …(3) again with respect to *x*,

$$\frac{d^2S}{dx^2}=4+\frac{8V}{x^3}=4+\frac{8x^3y}{x^3}=4+\frac{8y}{x}$$
$$\Rightarrow \left[\frac{d^2S}{dx^2}\right]_{y=x}=12>0$$

Therefore, S is minimum when length, breadth and height all are equal to *x*, i.e., when it is a cube.

Question 27:

Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4).

OR

Find the area enclosed between the parabola $4y=3x^2$ and the straight line $3x-2y+12=0$.

Solution:

Given, vertices of the triangle ABC are A (4,1), B(6,6) and C(8,4).

The equation of line AB is

$$y-1=\left(\frac{6-1}{6-4}\right)(x-4)$$

$$\Rightarrow 2y-2=5x-20$$

$$\Rightarrow y=\frac{5x-18}{2} \qquad ....(1)$$

The equation of line BC is

$$y-6=\left(\frac{4-6}{8-6}\right)(x-6)$$

$$\Rightarrow y=12-x \qquad ....(2)$$

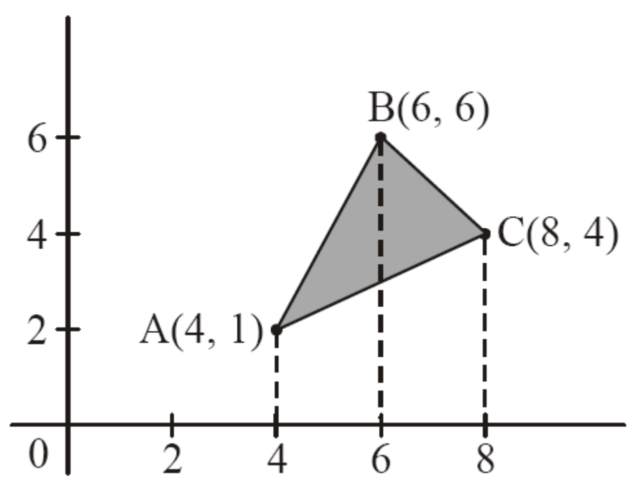
The equation of line CA is

$$y-4=\left(\frac{1-4}{4-8}\right)(x-8)$$

$$\Rightarrow 4y=3x-8$$

$$\Rightarrow y=\frac{3x-8}{4} \qquad ....(3)$$

The region bounded by these lines is shown below:



Required area = Area of the shaded region or Area of ΔABC

$$=\int_4^6\left(\frac{5x-18}{2}\right)dx+\int_6^8(12-x)\,dx-\int_4^8\left(\frac{3x-8}{4}\right)dx$$
$$=\int_4^6\left(\frac{5x}{2}-9\right)dx+\int_6^8(12-x)\,dx-\int_4^8\left(\frac{3x}{4}-2\right)dx$$

$$=\left[\frac{5x^2}{4}-9x\right]_4^6+\left[12x-\frac{x^2}{2}\right]_6^8-\left[\frac{3x^2}{8}-2x\right]_4^8=7+10-10=7 \text{ square units}$$

OR

$$4y = 3x^2 \quad ....(1)$$
$$3x - 2y + 12 = 0 \quad ...(2)$$

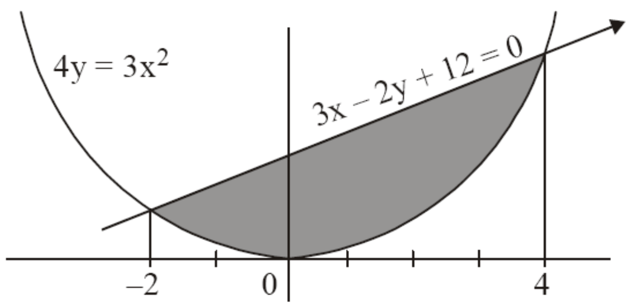
The curve … (1) represents an upward opening parabola having vertex at the origin, axis along the positive direction of y-axis. The curve … (2) represents a straight line. Solving …. (1) and ….(2), we have

$$3x - 2\left(\frac{3x^2}{4}\right) + 12 = 0$$
$$\Rightarrow 6x - 3x^2 + 24 = 0$$
$$\Rightarrow x^2 - 2x - 8 = 0$$
$$\Rightarrow (x+2)(x-4) = 0$$
$$\Rightarrow x = -2, 4$$

When $x = -2$, $y = 4$ when $x = 4$, $y = 12$

So, the point of intersection of the curves is (–2, 3) and (4, 12).

The area bounded by the given curves is shown below



Required area = Area of the shaded region

$$= \int_{-2}^{4} \left(\frac{3x+12}{2} - \frac{3}{4}x^2\right) dx$$
$$= \left(\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right)_{-2}^{4}$$
$$= \left(\frac{3}{4}\times 16 + 6\times 4 - \frac{64}{4}\right) - \left(\frac{3}{4}\times 4 + 6\times(-2) + \frac{8}{4}\right)$$
$$= 20 - (-7)$$

= 27 square units.

Question 28:

Find the particular solution of the differential equation $\left(x-y\right)\frac{dy}{dx}=\left(x+2y\right)$, given that $y = 0$ when $x = 1$.

Solution:

Given,

$$\left(x-y\right)\frac{dy}{dx}=x+2y$$
$$\Rightarrow \frac{dy}{dx}=\frac{x+2y}{x-y}$$

This is a homogenous differential equation

Putting, $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, we have,

$$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1+2v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+2x}{1-v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+2v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{1+v+v^2}dv=\frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{1-v}{1+v+v^2}\,dv = \int \frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{-(v-1)}{1+v+v^2}\,dv = \int \frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\int \frac{2v-2}{v^2+v+1}\,dv = \int \frac{-dx}{x}$$
$$\Rightarrow \int \frac{(2v+1)-3}{v^2+v+1}\,dv = -\int \frac{2dx}{x}$$

$$\Rightarrow \int \frac{(2v+1)}{v^2+v+1}\,dv - \int \frac{3}{v^2+v+1}\,dv = -\int \frac{2dx}{x} \qquad ...(1)$$

Suppose,

$$I_1 = \int \frac{(2v+1)}{v^2+v+1}\,dv \text{ and } I_2 = \int \frac{3}{v^2+v+1}\,dv$$

First we will solve I_1

$$I_1 = \int \frac{2v+1}{v^2+v+1}\,dv$$

Let

$$v^2+v+1=t$$
$$\Rightarrow (2v+1)\,dv = dt$$
$$\Rightarrow dv = \frac{dt}{(2v+1)}$$
$$\therefore I_1 = \int \frac{2v+1}{v^2+v+1}\,dv$$
$$\Rightarrow I_1 = \int \frac{dt}{t}$$
$$\Rightarrow I_1 = \log|t|$$
$$\Rightarrow I_1 = \log\left|v^2+v+1\right|$$

Now, we will calculate the value of I_2

$$I_2 = \int \frac{3}{v^2+v+1}\,dv$$

$$\Rightarrow I_2 = \int \frac{3}{v^2+2\times v\times\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$$

$$\Rightarrow I_2 = 3\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dv$$

$$\Rightarrow I_2 = 3\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\Rightarrow I_2 = 2\sqrt{3}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

So, the solution of ….(1) is given by

$$\log\left|v^2+v+1\right|-2\sqrt{3}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=-2\log|x|+c$$

Putting the value of $v=\frac{y}{x}$ in the above equation, we have

$$\log\left|x^2+y^2+xy\right|-2\log|x|=-2\log|x|+2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{x\sqrt{3}}\right)+c$$

$$\log\left|x^2+y^2+xy\right|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{x\sqrt{3}}\right)+c \quad ....(2)$$

It is given that $y = 0$ when $x = 1$.

$$\therefore \log|1| = 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)+c$$

$$\Rightarrow 2\sqrt{3}\left(\frac{\pi}{6}\right)+c=0$$

$$\Rightarrow c=-\frac{\pi}{\sqrt{3}}$$

Putting, $c=-\frac{\pi}{\sqrt{3}}$ in (2), we get

$$\log\left|x^2+y^2+xy\right|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{x\sqrt{3}}\right)-\frac{\pi}{\sqrt{3}}$$

Hence, $\log\left|x^2+y^2+xy\right|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{x\sqrt{3}}\right)-\frac{\pi}{\sqrt{3}}$, is the required solution.

Question 29:

Find the coordinates of the point where the line through the points (3, – 4, – 5) and (2, – 3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, – 3) and (0, 4, 3).

OR

A variable plane which remains at a constant distance 3*p* from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.

Solution:

Cartesian equation of a line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Equation of a line passing through (3, –4, –5) and (2, –3, 1) is

$$\frac{x-1}{2-3}=\frac{y-(-4)}{-3-(-4)}=\frac{z-(-5)}{1-(-5)}$$

$$\Rightarrow\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}$$

Now we will find the coordinates of any point on this line, this is given by

$$\frac{x-3}{-1}=\frac{y+4}{1}=\frac{x+5}{6}=k$$

$\Rightarrow x=3-k, y=k-4, z=6k-5$ where k is a constant

Let $M(3-k, k-4, 6k-5)$ be the required point of intersection

Now,

Let the equation of a plane passing through (1, 2, 3) be

$$a(x-1)+b(y-2)+c(z-3)=0 \quad ...(1)$$

Here, *a*, *b*, *c* are the direction ratios of the normal to the plane.

As the plane …(1) passes through (4, 2, –3), so

$$a(4-1)+b(2-2)+c(-3-3)=0$$
$$\Rightarrow 3a-6c=0 \quad ...(2)$$

Also, the plane …(1) passes through (0, 4, 3), so

$$a(0-1)+b(4-2)+c(3-3)=0$$
$$\Rightarrow -a+2b=0 \quad ....(3)$$

Solving …(2) and …(3) using the method of cross multiplication, we have

$$\frac{a}{0+12}=\frac{b}{6-0}=\frac{c}{6+0}$$
$$\Rightarrow \frac{a}{12}+\frac{b}{6}=\frac{c}{6}$$
$$\Rightarrow \frac{a}{2}=b=c=\lambda(say)$$
$$\Rightarrow a=2\lambda, b=\lambda, c=\lambda$$

From …(1), we get

$$2\lambda(x-1)+\lambda(y-2)+\lambda(z-3)=0$$
$$\Rightarrow 2x+y+z-7=0 \quad ...(4)$$

Putting $x=3-k, y=k-4, z=6k-5$ in …(4), we have

$$\Rightarrow 2(3-k)+(k-4)+(6k-5)-7=0$$
$$\Rightarrow 5k-10=0$$
$$\Rightarrow k=2$$

Putting $k=2$ in $M(3-k,k-4,6k-5)$, we get

$$M(3-k,k-4,6k-5)=$$
$$\Rightarrow M(3-2,2-4,6\times 2-5)$$
$$\Rightarrow M(1,-2,7)$$

Thus, the coordinates of the required point are (1, –2, 7).

OR

Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ....(1)$$

Where a, b and c are variables.

Let the plane meets the x, y and z axes at A (a, 0, 0), B (0, b, 0) and C (0, 0, c).

Suppose the coordinates of the centroid of triangle ABC be (α, β, γ)

$$\therefore \alpha=\frac{a+0+0}{3}=\frac{a}{3}, \beta=\frac{0+b+0}{3}=\frac{b}{3}, \gamma=\frac{0+0+c}{3}=\frac{c}{3}$$
$$\Rightarrow a=3\alpha, b=3\beta, c=3\gamma$$

It is given that the plane …(1) is at a distance of 3p from the origin.

$\therefore 3p=$ Length of perpendicular from (0, 0, 0) to the plane …(1)

$$\Rightarrow 3p=\frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$
$$\Rightarrow 3p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$
$$\Rightarrow \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=\frac{1}{3p}$$

Squaring on both sides, we get

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{9p^2} \qquad ...(2)$$

Substituting the values of a, b and c in …(2), we get

$$\frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2}=\frac{1}{p^2}$$

$$\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{p^2}$$

Replacing α, β and γ by x, y and z we have, the required locus of the centroid of the triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.