Solved Question Paper
2016
Class – XII
Subject – Mathematics
Delhi: Set – II

Time allowed: 3 hours **Maximum Marks: 100**

General Instructions:
i) All questions are compulsory.
ii) Please check that this Question Paper contains 26 Questions.
iii) Questions 1-6 in Section A are very short-answer type questions carrying 1 mark each.
iv) Questions 7-19 in Section B are long-answer I type questions carrying 4 marks each.
v) Questions 20-26 in Section C are long-answer II type questions carrying 6 marks each.
vi) Please write down the serial number of the questions before attempting it.

SECTION A

Question1. Matrix $A=\begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}$ is given to be symmetric, find values of a and b.

Solution:
We have,

$$A=\begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}$$

$$\text{Transpose of the matrix, } A'=\begin{bmatrix}0 & 3 & 3a\\ 2b & 1 & 3\\ -2 & 3 & -1\end{bmatrix}$$

Since, A is a symmetric matrix
Therefore, $A=A'$

$$\Rightarrow \begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}=\begin{bmatrix}0 & 3 & 3a\\ 2b & 1 & 3\\ -2 & 3 & -1\end{bmatrix}$$

$$\Rightarrow 2b=3 \text{ and } 3a=-2$$

$$\Rightarrow b=\frac{3}{2} \text{ and } a=-\frac{2}{3}.$$

Question2. Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2:1.

Solution:
Let P be any point that divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2:1.
Position vector of P are:

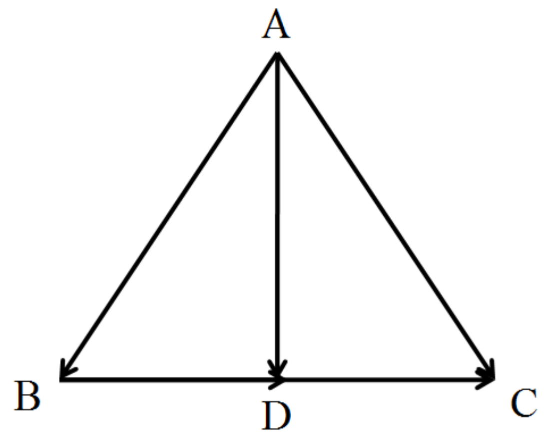
$$=\frac{2\left(2\vec{a}+\vec{b}\right)-1\left(\vec{a}-2\vec{b}\right)}{2-1}$$

$$=4\vec{a}+2\vec{b}-\vec{a}+2\vec{b}$$

$$=3\vec{a}+4\vec{b}.$$

Question3. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represents the two sides AB and AC, respectively of a triangle ABC. Find the length of median through A.

Solution:
Let ABC be the triangle as shown in the figure given below,



$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}$$

$$\overrightarrow{BC}=\left(3\hat{i}-\hat{j}+4\hat{k}\right)-\left(\hat{j}+\hat{k}\right)$$

$$\overrightarrow{BC}=3\hat{i}-2\hat{j}+3\hat{k}$$

$$\text{Now, D is the midpoint, } \therefore \overrightarrow{BD}=\frac{1}{2}\left(\overrightarrow{BC}\right)$$

$$\overrightarrow{BD}=\frac{1}{2}\left(3\hat{i}-2\hat{j}+3\hat{k}\right)$$

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$$

$$\overrightarrow{AD}=\hat{j}+\hat{k}+\frac{1}{2}\left(3\hat{i}-2\hat{j}+3\hat{k}\right)$$

$$\overrightarrow{AD}=\frac{3}{2}\hat{i}+\frac{5}{2}\hat{k}$$

$$\left|\overline{AD}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2}$$

$$\left|\overline{AD}\right| = \frac{1}{2}\sqrt{34} \text{ units.}$$

Question4. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Solution:

We have

Normal vector, $\hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$,

Perpendicular distance, $d = 5$

Vector equation of plane is given by $\vec{r} \cdot \hat{n} = d$ where, d is the perpendicular of the plane from the origin and $\hat{n}$ is the normal vector.

Therefore, equation of the plane is

$$\Rightarrow \vec{r} \cdot \left(2\hat{i} - 3\hat{j} + 6\hat{k}\right) = 5.$$

Question5. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$.

Solution:

Let, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and Applying $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$$

Exapnding along frist column, we have,

$$\Delta = \sin\theta\cos\theta$$

$$\Delta = \frac{1}{2}\sin 2\theta$$

Since, maximum value of sin can be 1,

Therefore, the maximum value of $\frac{1}{2}\sin 2\theta$ is $\frac{1}{2} \times 1 = \frac{1}{2}$.

Question6. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A-I)^3 + (A+I)^3 - 7A$.

Solution:

We have,

On simplifying we have,

$$\begin{aligned}(A-I)^3 + (A+I)^3 - 7A &= A^3 - I^3 - 3A^2I + 3I^2A + A^3 + I^3 + 3A^2I + 3I^2A - 7A \\ &= 2A^3 + 6I^2A - 7A \\ &= 2A^2 \cdot A + 6A - 7A \\ &= 2I \cdot A - A \\ &= 2A - A \\ &= A.\end{aligned}$$

SECTION B

Question7. Show that the equation of normal at any point t on the curve $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4\left(y\cos^3 t - x\sin^3 t\right) = 3\sin 4t$.

Solution:

We have given

$x = 3\cos t - \cos^3 t$; $y = 3\sin t - \sin^3 t$

Differentiating both sides with respect to t,

$$\Rightarrow \frac{dx}{dt} = -3\sin t + 3\cos^2 t\sin t$$

$$\Rightarrow \frac{dx}{dt} = -3\sin t\left(1 - \cos^2 t\right)$$

$$\Rightarrow \frac{dx}{dt} = -3\sin^3 t$$

Similarly

$$y = 3\sin t - \sin^3 t$$

$$\frac{dy}{dt} = 3\cos t - 3\sin^2 t\cos t$$

$$\frac{dy}{dt} = 3\cos t\left(1 - \sin^2 t\right)$$

$$\frac{dy}{dt} = 3\cos^3 t$$

Therefore

$$\frac{dy}{dx} = \left[\frac{\left(dy/dt\right)}{\left(dx/dt\right)}\right] = \frac{3\cos^3 t}{-3\sin^3 t}$$

$$\frac{dy}{dx} = -\frac{\cos^3 t}{\sin^3 t}$$

Slope of normal $= \left(-\frac{dx}{dy}\right) = \frac{\sin^3 t}{\cos^3 t}$

Therefore equation of normal is given by

$$\left(y-3\sin t+\sin^3 t\right)=\frac{\sin^3 t}{\cos^3 t}\left(x-3\cos t+\cos^3 t\right)$$
$$\cos^3 t\left(y-3\sin t+\sin^3 t\right)=\sin^3 t\left(x-3\cos t+\cos^3 t\right)$$
$$\Rightarrow y\cos^3 t-3\sin t\cos^3 t+\sin^3 t\cos^3 t=x\sin^3 t-3\sin^3 t\cos t+\sin^3 t\cos^3 t$$
$$\Rightarrow y\cos^3 t-x\sin^3 t=3\sin t\cos^3 t-3\sin^3 t\cos t$$
$$\Rightarrow y\cos^3 t-x\sin^3 t=3\sin t\cos t\left(\cos^2 t-\sin^2 t\right)$$
$$\Rightarrow y\cos^3 t-x\sin^3 t=\frac{3}{2}\sin 2t\left(\cos 2t\right)$$
$$\Rightarrow y\cos^3 t-x\sin^3 t=\frac{3}{4}\sin 4t$$
$\Rightarrow 4\left(y\cos^3 t-x\sin^3 t\right)=3\sin 4t$, which is the equation of normal at any point t on the curve.

Question8. Find $\int\frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$.

Solution:

$$\int\frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$$
$$=\int\frac{(3\sin\theta-2)\cos\theta}{5-\left(1-\sin^2\theta\right)-4\sin\theta}d\theta$$
$$=\int\frac{(3\sin\theta-2)\cos\theta}{5-\left(1-\sin^2\theta\right)-4\sin\theta}d\theta$$
Let, $\sin\theta=t\Rightarrow\cos\theta\,d\theta=dt$

Then,
$$\Rightarrow\int\frac{(3\sin\theta-2)\cos\theta}{5-\left(1-\sin^2\theta\right)-4\sin\theta}d\theta$$
$$=\int\frac{(3t-2)}{5-\left(1-t^2\right)-4t}dt$$
$$=\int\frac{(3t-2)}{t^2-4t+4}dt$$
$$=\int\frac{(3t-2)}{(t-2)^2}dt$$
By partial fraction,
$$\frac{(3t-2)}{(t-2)^2}=\frac{A}{(t-2)}+\frac{B}{(t-2)^2}$$
$$(3t-2)=A(t-2)+B$$
$$(3t-2)=At-2A+B$$
Comparing like terms,
$$A=3$$
$$-2A+B=-2$$
$$-2\times3+B=-2$$
$$B=-2+6$$
$$B=4$$

Therefore,
$$\frac{(3t-2)}{(t-2)^2}=\frac{3}{(t-2)}+\frac{4}{(t-2)^2}$$
$$\Rightarrow\int\frac{(3t-2)}{(t-2)^2}dt=\int\frac{3}{(t-2)}dt+\int\frac{4}{(t-2)^2}dt$$
$$\Rightarrow\int\frac{(3t-2)}{(t-2)^2}dt=3\ln(t-2)-\frac{4}{(t-2)}+c$$
$$\Rightarrow\int\frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta=3\ln\left|\sin\theta-2\right|-\frac{4}{(\sin\theta-2)}+c.$$

OR

Question8. Evaluate: $\int_0^{\pi}e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$

Solution:

We have given
$$\int_0^{\pi}e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$$
This can be simplified as,

$$I=\int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$$

$$I=\left[\sin\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{2}\right]_0^{\pi}-\int_0^{\pi}\cos\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{2}dx$$

$$I=\left[\sin\left(\frac{\pi}{4}+\pi\right)\frac{e^{2\pi}}{2}-\sin\left(\frac{\pi}{4}+0\right)\frac{e^{0}}{2}\right]-\int_0^{\pi}\cos\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{2}dx$$

$$I=\left[\sin\left(\frac{5\pi}{4}\right)\frac{e^{2\pi}}{2}-\sin\left(\frac{\pi}{4}\right)\frac{1}{2}\right]-\left[\cos\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{4}\right]_0^{\pi}-\int_0^{\pi}\sin\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{4}dx$$

$$I=\left[-\frac{1}{\sqrt{2}}\frac{e^{2\pi}}{2}-\frac{1}{2\sqrt{2}}\right]-\left[\cos\left(\frac{\pi}{4}+\pi\right)\frac{e^{2\pi}}{4}-\cos\left(\frac{\pi}{4}+0\right)\frac{e^{0}}{4}\right]-\frac{I}{4}$$

$$5\frac{I}{4}=-\left[\frac{e^{2\pi}+1}{2\sqrt{2}}\right]-\left[-\frac{1}{\sqrt{2}}\frac{e^{2\pi}}{4}-\frac{1}{\sqrt{2}}\frac{1}{4}\right]$$

$$I=-\frac{4}{5}\times\left[\frac{e^{2\pi}+1}{2\sqrt{2}}\right]+\frac{4}{5}\times\left[\frac{e^{2\pi}+1}{4\sqrt{2}}\right]$$

$$I=\frac{-\left(e^{2\pi}+1\right)}{5\sqrt{2}}$$

$$I=-\frac{\left(e^{2\pi}+1\right)}{5\sqrt{2}}.$$

Question9. Find: $\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx.$

Solution:

We have

$$\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$$

$$=\int\frac{\sqrt{x}}{\sqrt{\left(a^{3/2}\right)^2-\left(x^{3/2}\right)^2}}dx$$

Let, $x^{3/2}=t\Rightarrow\frac{3}{2}x^{1/2}dx=dt$

$$\sqrt{x}dx=\frac{2}{3}dt$$

Therefore,

$$\int\frac{\sqrt{x}}{\sqrt{\left(a^{3/2}\right)^2-\left(x^{3/2}\right)^2}}dx$$

$$=\frac{2}{3}\int\frac{dt}{\sqrt{\left(a^{3/2}\right)^2-t^2}}$$

$$=\frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}}+c$$

$$=\frac{2}{3}\sin^{-1}\frac{x^{3/2}}{a^{3/2}}+c$$

$$=\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+c.$$

Question10. Evaluate: $\int_{-1}^{2}\left|x^3-x\right|dx.$

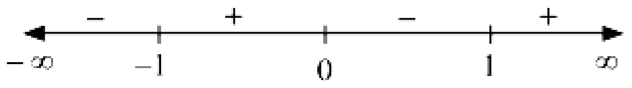
Solution:

We have

$f(x)=\left(x^3-x\right)$

$\Rightarrow f(x)=x(x+1)(x-1)$

Turning points of $f(x)$ are: –1, 0 and 1.



Therefore, between –1 to 2,

$f(x)>0$ for all $x\in(-1,0)\cup(-1,2)$

$f(x)<0$ for all $x\in(0,1)$

$$\int_{-1}^{2}\left|x^3-x\right|dx$$

$$=\int_{-1}^{0}\left|x^3-x\right|dx+\int_{0}^{1}\left|x^3+x\right|dx+\int_{1}^{2}\left|x^3-x\right|dx$$

$$=\int_{-1}^{0}\left(x^3-x\right)dx-\int_{0}^{1}\left(x^3-x\right)dx+\int_{1}^{2}\left(x^3-x\right)dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}-\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{0}^{1}+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{1}^{2}$$

$$=-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}+\frac{1}{2}+4-2-\frac{1}{4}+\frac{1}{2}$$

$$=\frac{3}{2}-\frac{3}{4}+2$$

$$=\frac{3}{4}+2$$

$$=\frac{11}{4}.$$

Question11. Find the particular solution of the following differential equation:
$\left(1-y^2\right)\left(1+\log x\right)dx+2xydy=0.$ Given that $y = 0$ at $x = 1$.

Solution:
We have
$$\left(1-y^2\right)\left(1+\log x\right)dx+2xydy=0$$
This can be written as,
$$\left(1-y^2\right)\left(1+\log x\right)dx+2xydy=0$$
$$\Rightarrow\left(1-y^2\right)\left(1+\log x\right)dx=-2xydy$$
$$\Rightarrow\frac{\left(1+\log x\right)}{x}dx=-2\frac{y}{\left(1-y^2\right)}dy$$
Integrating both sides as,
$$\int\frac{\left(1+\log x\right)}{x}dx=-2\int\frac{y}{\left(1-y^2\right)}dy$$
$$\Rightarrow\frac{\left(1+\log x\right)^2}{2}=\log\left(1-y^2\right)+c.............(1)$$
We have $y=0$ at $x=1$, therefore
$$\Rightarrow\frac{\left(1+\log 1\right)^2}{2}=\log\left(1-0\right)+c$$
$$\Rightarrow\frac{1}{2}=0+c$$
$$\Rightarrow c=\frac{1}{2}$$
Substituting $c=\frac{1}{2}$ in equation.....(1), we get
$$\frac{\left(1+\log x\right)^2}{2}=\log\left(1-y^2\right)+c$$
$$\Rightarrow\frac{\left(1+\log x\right)^2}{2}=\log\left(1-y^2\right)+\frac{1}{2}$$
$$\Rightarrow\frac{\left(1+\log x\right)^2}{2}=\frac{2\log\left(1-y^2\right)+1}{2}$$
$$\Rightarrow\left(1+\log x\right)^2=\log\left(1-y^2\right)^2+1.$$

Question12. Find the general solution of the following differential equation.
$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

Solution:
We have,
$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$
This can be written as,
$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$
$$\Rightarrow\frac{dy}{dx}=-\frac{(1+y^2)}{(x-e^{\tan^{-1}y})}$$
$$\Rightarrow\frac{dx}{dy}=-\frac{(x-e^{\tan^{-1}y})}{(1+y^2)}$$
$$\Rightarrow\frac{dx}{dy}+\frac{1}{(1+y^2)}x=\frac{e^{\tan^{-1}y}}{(1+y^2)}$$
This is in the form of linear differential equation $\frac{dx}{dy}+Px=Q$

Here,
$$P=\frac{1}{(1+y^2)};Q=\frac{e^{\tan^{-1}y}}{(1+y^2)}$$
Therefore integrating factor is:
$$I.F=e^{\int\frac{1}{(1+y^2)}dy}$$
$$\Rightarrow I.F=e^{\tan^{-1}y}$$
Now, solution of differential equation is:
$$x\times I.F=\int Q\times I.Fdy$$
$$x\times e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{(1+y^2)}\times e^{\tan^{-1}y}dy$$
$$x\times e^{\tan^{-1}y}=\int\frac{e^{2\tan^{-1}y}}{(1+y^2)}dy$$
Let,
$$\tan^{-1}y=t$$
$$\Rightarrow\frac{1}{1+y^2}dy=dt$$
Therefore,
$$x\times e^{\tan^{-1}y}=\int\frac{e^{2\tan^{-1}y}}{(1+y^2)}dy$$
$$x\times e^{\tan^{-1}y}=\int e^{2t}dt$$
$$x\times e^{\tan^{-1}y}=\frac{1}{2}e^{2t}+c$$
$$2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+C.$$

Question13. Show that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.

Solution:

We have three vectors: $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$.

Three vectors are coplanar, if their scalar triple product will be zero.

This implies,

$$\left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]=0$$

$$\Rightarrow \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right)\right]=0$$

$$\Rightarrow \left(\vec{a}+\vec{b}\right)\cdot\left[\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{a}\right)+0+\left(\vec{c}\times\vec{a}\right)\right]=0$$

$$\Rightarrow \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)+\vec{b}\cdot\left(\vec{b}\times\vec{c}\right)+\vec{a}\cdot\left(\vec{b}\times\vec{a}\right)+\vec{b}\cdot\left(\vec{b}\times\vec{a}\right)+\vec{a}\cdot\left(\vec{c}\times\vec{a}\right)+\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)+0+0+0+0+\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=0$$

$$\Rightarrow \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)+\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=0 \qquad \left[\text{since } \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)\right]$$

$$\Rightarrow 2\left[\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right]=0$$

$$\Rightarrow 2[a\ b\ c]=0$$

$$\Rightarrow [a\ b\ c]=0$$

Therefore, vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Question14. Find the vector and Cartesian equation of the line through the point (1, 2, –4) and perpendicular to the two lines.

$\vec{r}=\left(8\hat{i}-19\hat{j}+10\hat{k}\right)+\lambda\left(3\hat{i}-16\hat{j}+7\hat{k}\right)$ and $\vec{r}=\left(15\hat{i}+29\hat{j}+5\hat{k}\right)+\mu\left(3\hat{i}+8\hat{j}-5\hat{k}\right)$.

Solution:

Given, the equations of the given lines are

$\vec{r}=\left(8\hat{i}-19\hat{j}+10\hat{k}\right)+\lambda\left(3\hat{i}-16\hat{j}+7\hat{k}\right)$ …………………(1)

and $\vec{r}=\left(15\hat{i}+29\hat{j}+5\hat{k}\right)+\mu\left(3\hat{i}+8\hat{j}-5\hat{k}\right)$………………(2)

Vector parallel to equation …(1) is: $\overrightarrow{n_1}=3\hat{i}-16\hat{j}+7\hat{k}$

Vector parallel to equation …(2) is: $\overrightarrow{n_2}=3\hat{i}+8\hat{j}-5\hat{k}$

Since the required line is perpendicular to the given lines

Therefore,

Normal $\vec{n}$ parallel to the required line will be perpendicular to both $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$

$$\vec{n}=\overrightarrow{n_1}\times\overrightarrow{n_2}$$

$$\Rightarrow \vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$$\Rightarrow \vec{n}=24\hat{i}+36\hat{j}+72\hat{k}$$

Thus the vector equation of the required line is:

$$\vec{r}=\left(1\hat{i}+2\hat{j}-4\hat{k}\right)+\gamma\left(24\hat{i}+36\hat{j}+72\hat{k}\right)$$

In Cartesian form

$$\frac{x-1}{24}=\frac{y-2}{36}=\frac{z+4}{72}$$

$$\Rightarrow \frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

Question15. Three persons A, B and C apply for a job of manager in a private company. Chances of their selection (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change does not takes place, Find the probability that it is due to the appointment of C.

Solution:

Let E_1, E_2, E_3 be the events of of selecting A, B and C as managers, respectively.

Then,

$$P\left(E_1\right)=\frac{1}{7}$$

$$P\left(E_2\right)=\frac{2}{7}$$

$$P\left(E_3\right)=\frac{4}{7}$$

Let A be the event showing that change is not taking place.

Then,

Probability of A does not introducing change = $P\left(\frac{A}{E_1}\right)=1-0.8=0.2$

Probability of B does not tintroducing change= $P\left(\frac{A}{E_2}\right)=1-0.5=0.5$

Probability of C does not taking place,$P\left(\frac{A}{E_3}\right)=1-0.3=0.7$

If the change does not takes place, then the probability that it is due to the appointment of C is:

$$P\left(\frac{E_3}{A}\right)=\frac{P\left(\frac{A}{E_3}\right)\times P(E_3)}{P\left(\frac{A}{E_1}\right)\times P(E_1)+P\left(\frac{A}{E_2}\right)\times P(E_2)+P\left(\frac{A}{E_3}\right)\times P(E_3)}$$

$$\Rightarrow P\left(\frac{E_3}{A}\right)=\frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2+\frac{2}{7}\times 0.5+\frac{4}{7}\times 0.7}$$

$$\Rightarrow P\left(\frac{E_3}{A}\right)=\frac{2.8}{0.2+1+2.8}$$

$$\Rightarrow P\left(\frac{E_3}{A}\right)=\frac{2.8}{4}=0.7$$

OR

Question15. A and B throw a pair of dice alternatively. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

Solution:

Total of 7 can be obtained from the following ways: (1,6), (6,1), (2,5), (5,2), (3,4), (4,3)

Probability of getting total of 7 = $\frac{6}{36}=\frac{1}{6}$

Probability of not getting total of 7 = 1 – $\frac{1}{6}$ = $\frac{5}{6}$.

Total of 10 on the dice can be obtained in following ways

(4, 6), (6, 4), (5, 5)

Probability of getting a total of 10 = $\frac{3}{36}=\frac{1}{12}$

Probability of not getting a total of 10 = $1-\frac{1}{12}=\frac{11}{12}$

A wins if he gets total of 7 in 1st, 3rd 5th…and so on throws

Probability of winning of A $=\left(\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{1}{6}\right)+........$

This forms an infinite G.P with first term, a = 1/6 and common ratio, $r=\frac{5}{6}\times\frac{11}{12}$

Therefore its sum is:

$$=\frac{a}{1-r}=\frac{\frac{1}{6}}{1-\frac{5}{6}\times\frac{11}{12}}=\frac{\frac{1}{6}}{\frac{17}{72}}$$

$$=\frac{12}{17}$$

Now the probability of winning of $\text{B}=1-(\text{Probability of winning A})=1-\frac{12}{17}=\frac{5}{17}$.

Question16. Prove that: $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

Solution:

We have given

$$\text{L.H.S}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$$

$$=\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{12}{35}}{\frac{34}{35}}\right)+\tan^{-1}\left(\frac{\frac{11}{24}}{\frac{23}{24}}\right)$$

$$=\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right)$$

$$=\tan^{-1}\left(\frac{325}{325}\right)$$

$$=\tan^{-1}1$$

$$=\frac{\pi}{4}$$

= R.H.S

OR

Question16. Solve for x**:** $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$.

Solution:

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$
$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(2\cdot\frac{1}{\sin x}\right)$$
$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$
$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$
$$\Rightarrow \cot x = 1$$
$$\Rightarrow x = \frac{\pi}{4}.$$

Question17. The monthly incomes of Aryan and Babban are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves Rs 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

Solution:

Suppose the monthly incomes of Aryan and Babban be $3x$ and $4x$ and their monthly expenditures are $5y$ and $7y$ respectively.

Since each saves Rs 15000 per month.

Monthly saving of Aryan: $3x - 5y = 15{,}000$ and monthly saving of Babban: $4x - 7y = 15{,}000$.

The above system of equations can be written in the matrix form as,

A X = B,

Where,

$$A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\text{Now } A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$$
$$\Rightarrow |A| = -21 + 20 = -1$$

$$Adj(A) = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$
$$A^{-1} = \frac{adj(A)}{|A|}$$
$$A^{-1} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$
$$X = A^{-1}B$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000-75000 \\ 60000-45000 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix}$$

Therefore monthly income of Aryan = 3×30000= Rs.90,000.

Therefore monthly income of Babban = 4×30000= Rs.1,20,000.

The value reflected from this problem is that we should save some part of our monthly income for future use.

Question18. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ *and* $x = \frac{\pi}{3}$.

Solution:

We have given

$$x = a\sin 2t(1+\cos 2t)$$
$$y = b\cos 2t(1-\cos 2t)$$

Therefore,

$$\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$$
$$\frac{dx}{dt} = 2a\cos 2t + 2a\cos^2 2t - 2a\sin^2 2t$$
$$\frac{dx}{dt} = 2a\cos 2t + 2a\cos 4t$$

Similarly

$$\frac{dy}{dt} = -2b\sin 2t(1-\cos 2t) + b\cos 2t(2\sin 2t)$$
$$\frac{dy}{dt} = -2b\sin 2t + 2b\sin 2t\cos 2t + 2b\sin 2t\cos 2t$$
$$\frac{dy}{dt} = -2b\sin 2t + b\sin 4t + b\sin 4t$$
$$\frac{dy}{dt} = -2b\sin 2t + 2b\sin 4t$$

This implies,

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-2b\sin 2t + 2b\sin 4t}{2a\cos 2t + 2a\cos 4t}$$

Therefore value of $\frac{dy}{dx}$ at $x=\frac{\pi}{4}$ is

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}}=\frac{-2b\sin 2\left(\frac{\pi}{4}\right)+2b\sin 4\left(\frac{\pi}{4}\right)}{2a\cos 2\left(\frac{\pi}{4}\right)+2a\cos 4\left(\frac{\pi}{4}\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}}=\frac{-2b(1)+2b(0)}{2a(0)+2a(-1)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}}=\frac{b}{a}$$

Similarly

Therefore value of $\frac{dy}{dx}$ at $x=\frac{\pi}{3}$ is.

$$\frac{dy}{dx}=\frac{-2b\sin 2t+2b\sin 4t}{2a\cos 2t+2a\cos 4t}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{3}}=\frac{-2b\sin 2\left(\frac{\pi}{3}\right)+2b\sin 4\left(\frac{\pi}{3}\right)}{2a\cos 2\left(\frac{\pi}{3}\right)+2a\cos 4\left(\frac{\pi}{3}\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{3}}=\frac{-2b\left(\frac{\sqrt{3}}{2}\right)+2b\left(-\frac{\sqrt{3}}{2}\right)}{2a\left(-\frac{1}{2}\right)+2a\left(-\frac{1}{2}\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{3}}=\frac{-2b\sqrt{3}}{-2a}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\frac{\pi}{3}}=\frac{b\sqrt{3}}{a}.$$

OR

Question18. If $y=x^x$, prove that $\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$

Solution:

We have given

$y=x^x$

Taking log of both sides

$\log y=x\log x$

Differentiating both sides $w.r.t\ x$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=1\cdot\log x+x\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\log x+1$$

$$\Rightarrow\frac{dy}{dx}=x^x(\log x+1)$$

Differentiating both sides again $w.r.t\ x$

$$\Rightarrow\frac{d^2y}{dx^2}=x^x\frac{d(\log x+1)}{dx}+(\log x+1)\frac{d(x^x)}{dx}$$

$$\Rightarrow\frac{d^2y}{dx^2}=x^x\cdot\frac{1}{x}+x^x(\log x+1)(\log x+1)$$

$$\Rightarrow\frac{d^2y}{dx^2}=x^{x-1}+x^x(\log x+1)^2$$

Simplifying L.H.S,

$$\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}$$

$$=x^{x-1}+x^x(\log x+1)^2-\frac{1}{x^x}\left(x^x(\log x+1)\right)^2-\frac{x^x}{x}$$

$$=x^{x-1}+x^x(\log x+1)^2-x^x(\log x+1)^2-x^{x-1}$$

$=0$

=R.H.S

Hence proved.

Question19. Find the values of p and q for which

$$f(x)=\begin{cases}\frac{1-\sin^3 x}{3\cos^2 x} & ,If\ x<\frac{\pi}{2}\\ p & ,If\ x=\frac{\pi}{2}\\ q\frac{(1-\sin x)}{(\pi-2x)^2} & ,If\ x>\frac{\pi}{2}\end{cases}$$

is continuous at $x=\frac{\pi}{2}$

Solution:

Since f(x) is continuous at $x=\frac{\pi}{2}$,

Therefore, $\lim_{x\to\frac{\pi}{2}^-}f(x)=f\left(\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}^+}f(x)$

Now,

$$\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin^3 x}{3\cos^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin^2 x)}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin x)(1+\sin x)}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{(1+\sin^2 x+\sin x)}{3(1+\sin x)}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{(1+\sin^2 x+\sin x)}{3(1+\sin x)}$$

$$=\lim_{h\to 0}\frac{\left[1+\sin^2(\pi/2-h)+\sin(\pi/2-h)\right]}{3\left[1+\sin(\pi/2-h)\right]}$$

$$=\frac{(1+1+1)}{3(1+1)}$$

$$=\frac{1}{2}$$

therefore,

$$p=\frac{1}{2}$$

Also,

$$\lim_{x\to\frac{\pi}{2}^+} q\frac{(1-\sin x)}{(\pi-2x)^2}$$

$$=\lim_{h\to 0} q\frac{\left(1-\sin\left(\frac{\pi}{2}+h\right)\right)}{\left(\pi-2\left(\frac{\pi}{2}+h\right)\right)^2}$$

$$=\lim_{h\to 0} q\frac{(1-\cos h)}{(-2h)^2}$$

$$=\lim_{h\to 0} q\frac{2\sin^2\frac{h}{2}}{4h^2}$$

$$=\lim_{h\to 0} q\frac{\sin^2\frac{h}{2}}{2h^2}$$

$$=\lim_{h\to 0} q\frac{\sin^2\frac{h}{2}}{2\frac{h^2}{4}\cdot 4}$$

$$=\frac{q}{8}\quad\left[\text{Since}, \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Now,

$$\frac{q}{8}=\frac{1}{2}$$

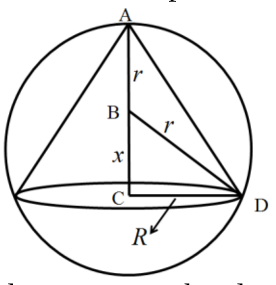
$$\Rightarrow q=4.$$

SECTION C

Question20. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $4r/3$. Also find maximum volume in terms of volume of sphere.

Solution:

Let the cone of greatest volume inscribed in a sphere of radius r as shown.



For maximum volume the axis of the cone must be along a diameter of the sphere.
Let AC be the axis of the cone and B be the centre of the sphere such that BC= x,
Then,
AC = AB + BC = $r + x$ = Height of the cone
Applying Pythagoras theorem in Δ BCD, we get

$$BD^2=BC^2+DC^2$$

$$r^2=x^2+R^2$$

$$R^2=r^2-x^2$$

Let V be the volume of the cone,
Then

$$V=\frac{1}{3}\pi R^2(r+x)$$

$$\Rightarrow V=\frac{1}{3}\pi R^2(r+x)$$

$$\Rightarrow V=\frac{1}{3}\pi(r^2-x^2)(r+x)$$

$$\Rightarrow \frac{dV}{dx}=\frac{1}{3}\pi(r^2-3x^2-2rx)$$

For critical points or maxima minima

$\frac{dV}{dx}=0$

$\Rightarrow \left(r^2-3x^2-2rx\right)=0$

$\Rightarrow x=\frac{r}{3}$

Now

$\frac{dV}{dx}=\frac{1}{3}\pi\left(r^2-3x^2-2rx\right)$

$\Rightarrow \frac{d^2V}{dx^2}=\frac{1}{3}\pi\left(-2r-6x\right)$

Putting $x=\frac{r}{3}$, in $\frac{d^2V}{dx^2}$,

$\frac{d^2V}{dx^2}=\frac{1}{3}\pi\left(-2r-6x\right)$

$\Rightarrow \frac{d^2V}{dx^2}=-\frac{4}{3}r\pi<0$

Thus, V is maximum if $x=\frac{r}{3}$.

Therefore altitude

AC = AB + BC = $r + x$

$=r+\frac{r}{3}$

$=\frac{4r}{3}$

Therefore volume of sphere is:

$V=\frac{1}{3}\pi\left(r^2-\frac{r^2}{9}\right)\left(r+\frac{r}{3}\right)$

$V=\frac{32\pi r^3}{81}$

$V=\frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$

Volume of the largest cone that can be inscribed in a sphere of radius r is $\frac{8}{27}$ of the volume of sphere.

OR

Question20. Find the intervals in which $f\left(x\right)=\sin 3x-\cos 3x$, $0<x<\pi$ is strictly increasing or strictly decreasing.

Solution:

We have given

$f\left(x\right)=\sin 3x-\cos 3x$

$\Rightarrow f'\left(x\right)=3(\cos 3x+\sin 3x)$

$f'\left(x\right)=0$

$\Rightarrow 3\left(\cos 3x+\sin 3x\right)=0$

$\Rightarrow \cos 3x=-\sin 3x$

$\Rightarrow \tan 3x=-1$

$\Rightarrow \tan 3x=\tan\left(\pi-\frac{\pi}{4}\right)$

$\Rightarrow \tan 3x=\tan\frac{3\pi}{4}$

$\Rightarrow 3x=n\pi\pm\frac{3\pi}{4}$

$\Rightarrow 3x=\frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4}.....$

$\Rightarrow x=\frac{\pi}{4},\frac{7\pi}{12},\frac{11\pi}{12},\frac{5\pi}{4}....$

Since $\frac{5\pi}{4}>\pi$ and we have to consider values only in the interval $0<x<\pi$

So, the turning points are $x=\frac{\pi}{4},\frac{7\pi}{12},\frac{11\pi}{12}$

So the intervals are $\left(0,\frac{\pi}{4}\right),\left(\frac{\pi}{4},\frac{7\pi}{12}\right),\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)\left(\frac{11\pi}{12},\pi\right)$

In the interval $\left(0,\frac{\pi}{4}\right)$, if we consider $x=\frac{\pi}{12}$

Then

$f'\left(\frac{\pi}{12}\right)=\cos\left(3\times\frac{\pi}{12}\right)+\sin\left(3\times\frac{\pi}{12}\right)$

$f'\left(\frac{\pi}{12}\right)=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}>0$

Therefore the function is strictly increasing in $\left(0,\frac{\pi}{4}\right)$

In the interval $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$, if we consider $x=\frac{\pi}{3}$

Then

$f'\left(\frac{\pi}{3}\right)=\cos\left(3\times\frac{\pi}{3}\right)+\sin\left(3\times\frac{\pi}{3}\right)$

$f'\left(\frac{\pi}{3}\right)=\cos\pi+\sin\pi=-1+0=-1<0$

Therefore the function is strictly decreasing in $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$

In the interval $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$, if we consider $x = \frac{2\pi}{3}$

Then

$$f'\left(\frac{2\pi}{3}\right) = \cos\left(3\times\frac{2\pi}{3}\right) + \sin\left(3\times\frac{2\pi}{3}\right)$$

$$f'\left(\frac{2\pi}{3}\right) = \cos\pi + \sin\pi = 1 + 0 = 1 > 0$$

Therefore the function is strictly increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

In the interval $\left(\frac{11\pi}{12}, \pi\right)$, if we consider $x = \frac{17\pi}{18}$

Then

$$f'(x) = \cos 3x + \sin 3x$$

$$f'(x) = \cos\left(3\times\frac{17\pi}{18}\right) + \sin\left(3\times\frac{17\pi}{18}\right)$$

$$f'(x) = \cos\left(3\pi - \frac{\pi}{6}\right) + \sin\left(3\pi - \frac{\pi}{6}\right)$$

$$f'(x) = -\frac{\sqrt{3}}{2} + \frac{1}{2} < 0$$

Therefore the function is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$

$x = \frac{\pi}{3}$

Then

$$f'(x) = \cos 3x + \sin 3x$$

$$f'(x) = \cos\left(3\times\frac{\pi}{3}\right) + \sin\left(3\times\frac{\pi}{3}\right)$$

$$f'(x) = \cos\pi + \sin\pi = -1 + 0 = -1 < 0$$

Therefore the function is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

In the interval $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$, if we consider

$x = \frac{2\pi}{3}$

Then

$$f'(x) = \cos 3x + \sin 3x$$

$$f'(x) = \cos\left(3\times\frac{2\pi}{3}\right) + \sin\left(3\times\frac{2\pi}{3}\right)$$

$$f'(x) = \cos\pi + \sin\pi = 1 + 0 = 1 > 0$$

Therefore the function is strictly increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

In the interval $\left(\frac{11\pi}{12}, \pi\right)$, if we consider

$x = \frac{17\pi}{18}$

Then

$$f'(x) = \cos 3x + \sin 3x$$

$$f'(x) = \cos\left(3\times\frac{17\pi}{18}\right) + \sin\left(3\times\frac{17\pi}{18}\right)$$

$$f'(x) = \cos\left(3\pi - \frac{\pi}{6}\right) + \sin\left(3\pi - \frac{\pi}{6}\right)$$

$$f'(x) = -\frac{\sqrt{3}}{2} + \frac{1}{2} < 0$$

Therefore the function is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$.

Question21. Using integration, find the area of the region $\{(x, y): x^2 + y^2 \le 2ax, y^2 \ge ax, y \ge 0\}$

Solution:

We have

$x^2 + y^2 \le 2ax$

$\Rightarrow x^2 - 2ax + y^2 \le 0$

$\Rightarrow x^2 - 2ax + a^2 - a^2 + y^2 \le 0$

$\Rightarrow (x-a)^2 + y^2 \le a^2.$

Also

$y^2 \ge ax$

To find the point of intersection of $(x-a)^2 + y^2 = a^2$ and $y^2 = ax$

Substituting $y^2 = ax$ in $(x-a)^2 + y^2 = a^2$,

$\Rightarrow (x-a)^2 + ax = a^2$

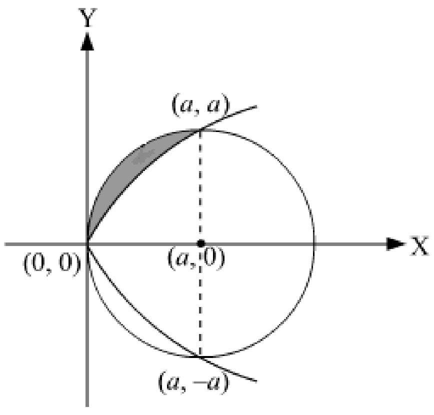
$\Rightarrow x^2 + a^2 - 2ax + ax = a^2$

$\Rightarrow x^2 - ax = 0$

$\Rightarrow x(x-a) = 0$

$\Rightarrow x = 0, a$

The graph is as follows:



Area of the shaded portion is:

$$\text{Area} = \int_0^a \left(\sqrt{a^2-(x-a)^2} - \sqrt{ax}\right)dx$$

$$\text{Area} = \left(\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} - \frac{2\sqrt{a}}{3}(x)^{3/2}\right)_0^a$$

$$\text{Area} = \left(0+0-0-\frac{a^2}{2}\sin^{-1}(-1) - \frac{2\sqrt{a}}{3}(a)^{3/2}\right)_0^a$$

$$\text{Area} = \frac{a^2}{2}\frac{\pi}{2} - \frac{2}{3}a^2$$

$$\text{Area} = a^2\left(\frac{\pi}{4}-\frac{2}{3}\right) \text{ square units.}$$

Question22. Find the coordinate of the point P where the line through A (3, –4, –5) and B (2, –3, 1) crosses the plane passing through three points L (2, 2, 1), M (3, 0, 1) and N (4, –1, 0). Also find the ratio in which P divides the line segment AB.

Solution:

The equation of the plane passing through three given points is given by:

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y-(-1) & z-0 \end{vmatrix} = 0$$

By elementary operations $R_2 \to R_1 - R_2$ and $R_3 \to R_1 - R_3$ we have,

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)=0$$

$$\Rightarrow 2x-4+y-2+z-1=0$$

$$\Rightarrow 2x+y+z-7=0$$

Now the equation of line passing through two given points is:

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$

$$x=-\lambda+3$$

$$y=\lambda-4$$

$$z=6\lambda-5$$

Since line intersects the plane, so these points must satisfy the equation of plane, Therefore

$$2x+y+z-7=0$$

$$\Rightarrow 2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7=0$$

$$\Rightarrow -2\lambda+6+\lambda-4+6\lambda-5-7=0$$

$$\Rightarrow 5\lambda-10=0$$

$$\Rightarrow \lambda=2$$

So the points of intersection P are

$$(-\lambda+3, \lambda-4, 6\lambda-5)$$

$$\Rightarrow (-2+3, 2-4, 6\times 2-5)$$

$$\Rightarrow (1,-2,7)$$

Let P divides the line segment AB in the ratio $m{:}n$.

Then,

$$1=\frac{2m+3n}{m+n}$$

$$\Rightarrow m=2n$$

$$\Rightarrow \frac{m}{n}=\frac{2}{1}$$

Therefore, P divides line segment AB in the ratio 2:1.

Question23. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution.

Solution:

Let X denote the total number of red balls when four balls are drawn one by one with replacement

Now, Probability of getting a red ball in one draw = 2/3

Probability of getting a red ball in one draw ==1/3

The probability distribution is as follows:

X	0	1	2	3	4
P(X)	${}^4C_0\left(\frac{1}{3}\right)^4$	${}^4C_1\frac{2}{3}\left(\frac{1}{3}\right)^3$	${}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	${}^4C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$	${}^4C_4\left(\frac{2}{3}\right)^4$
	1/81	8/81	24/81	32/81	16/81

Therefore mean is:

$$Mean\left(\overline{X}\right)=0\times\frac{1}{81}+1\times\frac{8}{81}+2\times\frac{24}{81}+3\times\frac{32}{81}+4\times\frac{16}{81}$$

$$Mean\left(\overline{X}\right)=\frac{1}{81}(8+48+96+64)$$

$$Mean\left(\overline{X}\right)=\frac{216}{81}$$

$$Mean\left(\overline{X}\right)=\frac{8}{3}$$

And

$$\text{Variance}=\sum P_iX_i^{\;2}-\sum\left(P_iX_i\right)^2$$

$$=\left(0^2\times\frac{1}{81}+1^2\times\frac{8}{81}+2^2\times\frac{24}{81}+3^2\times\frac{32}{81}+4^2\times\frac{16}{81}\right)-\left(\frac{8}{3}\right)^2$$

$$=\frac{648}{81}-\frac{64}{9}$$

$$=\frac{8}{9}.$$

Question24. A manufacturer produces two products A and B .Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each product of B requires 2 hrs on first machine and 1 hr on second machine. Each unit of product A is sold at Rs 7 profit and That of B at a profit of Rs 4. Find the production level per day for maximum profit graphically.

Solution:

Let number of units of A manufactured be x

Let number of units of B manufactured be y

Total profit is given by: Z = $7x+4y$

We have to maximize Z=$7x+4y$ subject to the constraints

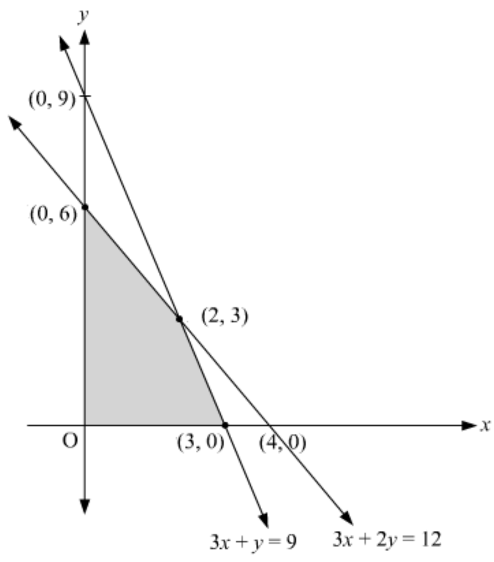
$3x+2y\le 12$ (for first machine)

$3x+y\le 9$ (for second machine)

$x\ge 0$

$y\ge 0$

Graphically, this can be expressed as,



Therefore;

CORNER POINT	Z=7x+4y
(0,6)	24
(2,3)	26
(3,0)	21
(0,0)	0

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B to obtain maximum profit.

Question25. Let f:N $\rightarrow$N be a function defined as $f(x)=9x^2+6x-5$. Show that f: N$\rightarrow$ S, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

Solution:

Given

$f\left(x\right)=9x^2+6x-5$

Let

$y=9x^2+6x-5$

$\Rightarrow y=\left(3x\right)^2+2\cdot 3x+1-1-5$

$\Rightarrow y=\left(\left(3x\right)^2+2\cdot 3x+1\right)-6$

$\Rightarrow y=\left(3x+1\right)^2-6$

$\Rightarrow y+6=\left(3x+1\right)^2$

$\Rightarrow \sqrt{y+6}=\left(3x+1\right)$

$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}$ where $x\in N$

Now, $\sqrt{y+6}-1>0$

$\Rightarrow y+6>1\Rightarrow y>-5$ also $y\in N$.

Therefore, the function will be invertible if the range of the given function is natural number.

Hence, inverse of the function $f(x)$ is $f^{-1}(y)$ and $f^{-1}(y)=\dfrac{\sqrt{y+6}-1}{3}$

Now

$$f^{-1}(y)=\frac{\sqrt{y+6}-1}{3}$$
$$f^{-1}(43)=\frac{\sqrt{43+6}-1}{3}$$
$$f^{-1}(43)=\frac{7-1}{3}$$
$$f^{-1}(43)=\frac{6}{3}=2$$

Similarly

$$f^{-1}(163)=\frac{\sqrt{163+6}-1}{3}$$
$$f^{-1}(163)=\frac{13-1}{3}$$
$$f^{-1}(163)=\frac{12}{3}=4.$$

Question26. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ is divisible by $(x+y+z)$ and hence find the quotient.

Solution:

We have given

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Performing $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \begin{vmatrix} xy+yz+zx-(x^2+y^2+z^2) & xy+yz+zx-(x^2+y^2+z^2) & xy+yz+zx-(x^2+y^2+z^2) \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Taking out $\left[xy+yz+zx-(x^2+y^2+z^2)\right]$ from R_1

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right]\begin{vmatrix} 1 & 1 & 1 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$

Performing $C_2 \to C_2-C_1; C_3 \to C_3-C_1$

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right]\begin{vmatrix} 1 & 0 & 0 \\ zx-y^2 & xy-zx-z^2+y^2 & yz-zx-x^2+y^2 \\ xy-z^2 & yz-xy-x^2+z^2 & zx-xy-y^2+z^2 \end{vmatrix}$$

Taking out common $(x+y+z)^2$

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right]\begin{vmatrix} 1 & 0 & 0 \\ zx-y^2 & (x+y+z)(y-z) & (x+y+z)(y-x) \\ xy-z^2 & (x+y+z)(z-x) & (x+y+z)(z-y) \end{vmatrix}$$

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right](x+y+z)^2\begin{vmatrix} 1 & 0 & 0 \\ zx-y^2 & (y-z) & (y-x) \\ xy-z^2 & (z-x) & (z-y) \end{vmatrix}$$

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right](x+y+z)^2\left[(y-z)(z-y)-(y-x)(z-x)\right]$$

$$\Rightarrow \left[xy+yz+zx-(x^2+y^2+z^2)\right]^2(x+y+z)^2$$

Therefore given determinant is divisible by $(x+y+z)$ as one of its factor is $(x+y+z)$

And the quotient after division will be:

$\left[xy+yz+zx-(x^2+y^2+z^2)\right]^2(x+y+z).$

OR

Question26. Using elementary transformations, find the inverse of the matrix A = $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and use it to solve the following system of linear equations.

$$8x+4y+3z=19$$
$$2x+y+z=5$$
$$x+2y+2z=7$$

Solution:

Let

$A = IA$

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying, $R_1 \Leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying, $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 8R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying, $R_2 \to \frac{R_2}{-3}$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/3 & 2/3 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying, $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 12R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 0 & -1/3 & 2/3 \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying, $R_3 \to -R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} A$$

Therefore,

$$A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equations is:

$8x + 4y + 3z = 19$

$2x + y + z = 5$

$x + 2y + 2z = 7$

This can be written as, $A X = B$

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$AX = B$

$\Rightarrow X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{14}{3} \\ -19 + 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = 1.$